LIE GROUPS AND LIE ALGEBRAS

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Chapitre 1

Linear Lie groups

A linear Lie group is a closed subgroup of $GL(n, \mathbb{R})$. To a linear Lie group one associates its Lie algebra. In this way the properties of the group are translated in terms of the linear algebra properties of its Lie algebra. We saw several examples in Section 1.3. Let us observe that is a line $GL(n, \mathbb{C})$ is a Lie group since it can be seen as a closed subgroup of $GL(n, \mathbb{R})$. In fact, to a matrix $Z = X + iY$ in $M(n, \mathbb{C})$ one associates the matrix
$\widetilde{Z} = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$ in $M(2n, \mathbb{C})$, and the map $Z \longmapsto \widetilde{Z}$ is an algebra morphism which maps $GL(n, \mathbb{C})$ onto a closed subgroup of $GL(n, \mathbb{R})$.

1.1 One parameter subgroups

Let G be a topological group. A one parameter subgroup of G is a continuous group morphism $\gamma : \mathbb{R} \longrightarrow G$, $\mathbb{R}$ being equipped with the additive group structure.

Théorème 1.1.1 *Let : $\gamma : \mathbb{R} \longrightarrow GL(n, \mathbb{R})$ be a one parameter subgroup of $GL(n, \mathbb{R})$. Then γ is C^∞ and $\gamma(t) = exp(tA)$, with $A = \gamma(0)$. In fact γ is even real analytic, as can be proved.*

Preuve. Assume that γ is C^1. Then

$$\begin{aligned}\gamma'(t) &= \lim_{s\to 0}\frac{\gamma(t+s)-\gamma(t)}{s}\\ &= \gamma(t)\lim_{s\to 0}\frac{\gamma(s)-\gamma(0)}{s}\\ &= \gamma(t)\gamma'(0)=\gamma'(0)\gamma(t)\end{aligned}$$

Put $A=\gamma'(0)$. Then $\gamma'(t)=A\gamma(t)$. This differential equation has a unique solution γ such that $\gamma(0)=1$, which is given by $\gamma(t)=\exp(tA)$ In fact, if γ is such a solution ■

$$\frac{d}{dt}(\exp(-tA)\gamma(t))=\exp(-tA)(\gamma'(t)-A\gamma(t))=0$$

We will now show that γ is C^1. Let α be a C^∞ function on $\mathbb{R}$ with compact support, and consider the regularised function f of γ :

$$f(t)=\int_{-\infty}^{+\infty}\alpha(t-s)\gamma(s)ds$$

Then $f:\mathbb{R}\longrightarrow M(n,\mathbb{R})$ is C^∞, and

$$f(t)=\int_{-\infty}^{+\infty}\alpha(s)\gamma(t-s)ds=\int_{-\infty}^{+\infty}\alpha(s)\gamma(-s)ds.\gamma(t).$$

We will choose the function α in such a way that the matrix

$$B=\int_{-\infty}^{+\infty}\alpha(s)\gamma(-s)ds$$

is invertible. It will followthat γ is C^∞. If $\|B-I\|<1$ then it holds. Let $\alpha\geq 0$, with integral equal to one. Then $\|B-I\|\leq f(t)=\int_{-\infty}^{+\infty}\alpha(s)\left\|\gamma(-s)-I\right\|ds$. Since γ is continuous at 0, for every

$\varepsilon>0$ there exists $\eta>0$ such that, if $|s|\leq\eta$, then $\|\gamma(-s)-I\|\leq\varepsilon$. If the support of α is contained in $[-\eta,\eta]$, then $\|B-I\|\leq\varepsilon$.

1.2 Lie algebra of a linear Lie group

Let G be a linear Lie group, that is a closed subgroup of $GL(n,\mathbb{R})$. We associate to the group G the set $g=Lie(G)=\{X\in M(n,\mathbb{R})|\forall t\in\mathbb{R}, exp(tX)\in G\}$.

Théorème 1.2.1 *(i) The set g is a vector subspace of $M(n,\mathbb{R})$.*

(ii) If $X, Y \in g$, then $[X,Y] := XY - YX \in \mathcal{G}$.

Preuve. (a) If $X, Y \in \mathcal{G}$, then $\left(\exp \frac{t}{k}X \exp \frac{t}{k}Y\right)^k \in G$, and, since G is closed, as $k \longrightarrow \infty$, $\exp(t(X+Y)) \in G$ by Corollary 2.2.4, hence $X + Y \in \mathcal{G}$. ■

(b) Similarly, for $t > 0$, $\lim\limits_{k\to\infty} \left(\exp \frac{\sqrt{t}}{k}X \exp \frac{\sqrt{t}}{k}Y \exp -\frac{\sqrt{t}}{k}X - \exp -\frac{\sqrt{t}}{k}Y\right)^{k^2} =$ $\exp(t\,[X,Y] \in G$, hence $[X,Y] \in \mathcal{G}$.

A real (respectively complex) Lie algebra is a vector space $\mathcal{G}$ over $\mathbb{R}$ (respectively C) equipped with a linear map $\mathcal{G}\times\mathcal{G} \longrightarrow \mathcal{G}$ by $(X,Y) \longmapsto [X,Y]$ called the bracket or commutator of X and Y , such that

(1) $[X,Y] = -[Y,X]$,

(2) $[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$.

Relation (2) is called the Jacobi identity.

The space $M(n,\mathbb{R})$ equipped with the product $[X,Y] = XY - YX$ is a Lie algebra. If $G \subset GL(n,\mathbb{R})$ is a linear Lie group, then $\mathcal{G} = Lie(G)$ is a subalgebra of $M(n,\mathbb{R})$, it is the Lie algebra of G.

Exemple 1.2.2 *$LieGL(n,\mathbb{R}) = M(n,\mathbb{R})$,*

$Lie\,[SL(n,\mathbb{R})] = \{X \in M(n,\mathbb{R}) | trX = 0\}$,

$Lie\,[SO(n)] = \left\{X \in M(n,\mathbb{R}) | X^T = -X\right\}$,

$Lie\,[Sp(n,\mathbb{R})] = \left\{\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \backslash A \in M(n,\mathbb{R}), B, C \in Sym(n,\mathbb{R})\right\}$,

$Lie\,[(U(n))] = \{X \in M(n,\mathbb{R}) | X^* = -X\}$

Consider $G = SL(2,\mathbb{R})$ and let $\mathcal{G} = sl(2,\mathbb{R})$ be its Lie algebra. The following matrices constitute a basis of $\mathcal{G}$:

$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $[H,E] = 2E$, $[H,F] = -2F$, $[E,F] = H$.

Let G be the group '$ax+b$', that is the group of affine linear transformations of $\mathbb{R}$. It is the set $\mathbb{R}^* \times \mathbb{R}$ equipped with the product $(a_1,b_1)(a_2,b_2) = (a_1a_2, a_1b_2 + b_1)$. This is not a group of matrices, but it can be identified with the closed subgroup of $GL(2,\mathbb{R})$ whose elements are the matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$.

The matrices $X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ constitute a basis of its Lie algebra and $[X_1, X_2] = X_2$. Let G be the motion group of $\mathbb{R}^2$, that is the group of affine linear transformations of the form $(x, y) \longmapsto (x\cos\theta - y\sin\theta + a, x\sin\theta + y\cos\theta + b)$. The group G can be identified with the subgroup of $GL(3, \mathbb{R})$ whose elements are the matrices $\begin{pmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{pmatrix}$

Its Lie algebra $\mathcal{G}$ has dimension 3. The following matrices constitute a basis

for g : $X_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$,

and $[X_1, X_2] = X_3, [X_1, X_3] = -X_2$, $[X_2, X_3] = 0$. Let $\mathcal{G}$ and $\mathcal{H}$ be two Lie algebras over $\mathbb{R}$ (or $\mathbb{C}$). A Lie algebra morphism of $\mathcal{G}$ into h is a linear map $A : \mathcal{G} \longrightarrow \mathcal{H}$satisfying

$[AX, AY] = A[X, Y]$. The group of automorphisms of the Lie algebra g is denoted by $Aut(\mathcal{G})$.

Let G be a linear Lie group, and $\mathcal{G} = Lie(G)$ its Lie algebra. By the definition of the Lie algebra of G, the exponential map maps g into G : exp : $\mathcal{G} \longrightarrow G$. For $y \in G, X \in \mathcal{G}, t \in \mathbb{R}$,

$y\exp(tX)y^{-1} = exp(tyXy^{-1})$. Hence $yXy^{-1} \in \mathcal{G}$. The map $Ad(y) : X \to Ad(y)X = yXy^{-1}$ is an automorphism of the Lie algebra $\mathcal{G}$, $Ad(y)[X, Y] = [Ad(y)X, Ad(y)Y]$, $(X, Y \in \mathcal{G})$.

Furthermore $Ad(y_1y_2) = Ad(y_1) \circ Ad(y_2)$, and this means that the map $Ad : G \longrightarrow Aut(\mathcal{G})$ is a group morphism.

Proposition 1.2.3 *(i) For* $X \in \mathcal{G}$, $\left(\frac{d}{dt}Ad(\exp(tX)\right)_{t=0} = adX$.

(ii) Let us denote by Exp the exponential map from $End(\mathcal{G})$ into $GL(\mathcal{G})$. Then $Exp(adX) = Ad(expX)$, $(X \in \mathcal{G})$.

Preuve. (a) $\left(\frac{d}{dt}Ad(\exp(tX)\right)_{t=0} = \left(\frac{d}{dt}\exp(tX)Y\exp(-tX)\right)_{t=0} = [X, Y]$. ■

(b) Put $\gamma_1(t) = Exp(tadX)$, $\gamma_2(t) = Ad(\exp(tX))$,They are two one parameter subgroups of $GL(\mathcal{G})$, and $\gamma_1'(0) = ad(X)$, $\gamma_2'(0) = ad(X)$ Therefore $\gamma_1(t) = \gamma_2(t)$ $(t \in \mathbb{R})$.

1.3 Linear Lie groups are submanifolds

Let us recall first the definition of a submanifold in a finite dimensional real vector space. A submanifold of dimension m in $\mathbb{R}^N$ is a subset M with the following property : for every $x \in M$ there exists a neighbourhood U of 0 in $\mathbb{R}^N$, a neighbourhood W of x in $\mathbb{R}^N$ and a diffeomorphism from U onto W such that $\Phi(U \cap \mathbb{R}^m) = W \cap M$.

Théorème 1.3.1 *Let G be a linear Lie group and $\mathcal{G} = Lie(G)$ be its Lie algebra.There exists a neighbourhood U of 0 in $\mathcal{G}$ and a neighbourhood V of I in G such that*

$\exp : U \longrightarrow V$ is a homeomorphism.

Preuve. Let $G \subset GL(n, \mathbb{R})$ be a linear Lie group, and $\mathcal{G} \subset M(n, \mathbb{R})$ be its Lie

algebra. Let U_0 be a neighbourhood of 0 in $M(n, \mathbb{R})$ and V_0 a neighbourhood of I in $GL(n, \mathbb{R})$ for which $\exp : U_0 \longrightarrow V_0$ is a diffeomorphism. Then $U_0 \cap \mathcal{G}$ is a neighbourhood of 0 in $\mathcal{G}$, the restriction of the exponential map to $U_0 \cap \mathcal{G}$ is injective and maps $U_0 \cap \mathcal{G}$ into $V_0 \cap G$, but one does not know yet whether $\exp(U_0 \cap \mathcal{G}) = V_0 \cap G$, even if one assumes that G is connected. ■

Lemme 1.3.2 *Let (g_k) be a sequence of elements in G which converges to I . One assumes that, for all k, $g_k = I$. Then the accumulation points of the sequence $X_k = \frac{\log g_k}{\|\log g_k\|}$*

belong to g.

Preuve. We may assume that $\lim_{k\to\infty} X_k = X \in M(n, \mathbb{R})$ Put $Y_k = \log g_k$ and, for $t \in \mathbb{R}$, $\lambda_k = \frac{t}{\|\log g_k\|}$, then $\exp(tX) = \lim_{k\to\infty} \exp(\lambda_k Y_k)$. ■

Let us denote by $[\lambda_k]$ the integer part of λ_k. We can write $\exp(\lambda_k Y_k) = (\exp(Y_k))^{[\lambda_k]} \exp((\lambda_k - [\lambda_k])Y_k)$, and $\|\lambda_k - [\lambda_k])Y_k\| \leq \|Y_k\| \longrightarrow 0$, hence, since $\exp Y_k = g_k$, $\exp(tX) = \lim_{k\to\infty} (g_k)^{[\lambda_k]} \in G$, and this proves that X belongs to $\mathcal{G}$.

Lemme 1.3.3 *Let $\mathcal{M}$ be a subspace of $M(n,\mathbb{R})$, complementary to $\mathcal{G}$. Then there exists a neighbourhood U of 0 in m such that* $\exp U \cap G = \{I\}$.

Preuve. Let us assume the opposite. In this case there exists a sequence $X_k \in m$ with limit 0 such that $g_k = \exp X_k$, $g_k \neq I$, $g_k \in G$. Let Y be an accumulation point of the sequence $\frac{X_k}{\|X_k\|}$. By Lemma $Y \in \mathcal{G} \cap \mathcal{M} = \{0\}$, and this is impossible since $\|Y\| = 1$. ■

Lemme 1.3.4 *Let E and F be two complementary subspaces in $M(n,\mathbb{R})$. Then the map $\Phi : E \times F \longrightarrow GL(n,\mathbb{R})$ by $(X,Y) \longmapsto \exp X \exp Y$. is differentiable, and $D\Phi_{(0,0)}(X,Y) = X + Y$*

The proof is left to the reader. We can now finish the proof of Theorem . Let m be a subspace of $M(n,\mathbb{R})$ complementary to g, and consider the map $\Phi : g \times m \longrightarrow GL(n,\mathbb{R})$ by $(X,Y) \longmapsto \exp X \exp Y$ There exists a neighbourhood U of 0 in $\mathcal{G}$, a neighbourhood V of 0 in m, and a neighbourhood W of I in $GL(n,\mathbb{R})$ such that the restriction of to$U \times V$ is a diffeomorphism onto W. Observe that $\exp U = \Phi(U \times \{0\}) \subset W \cap G$. By Lemma the neighbourhood V can be chosen such that $\exp V \cap G = \{I\}$. Let us show that $\exp U = W \cap G$. Let $g \in W \cap G$. One can write $g = \exp X \exp Y$ $(X \in U, Y \in V)$, and then $\exp Y = \exp(-X)g \in \exp V \cap G = \{I\}$, hence $g = \exp X$.

Corollaire 1.3.5 *Alinear Lie group $G \subset GL(n,\mathbb{R})$ is a submanifold of $M(n,\mathbb{R})$ of dimension $m = dimg$.*

Preuve. Let $g \in G$ and let $L(g)$ be the map $L(g) : GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$, by $h \longmapsto gh$. Let U be a neighbourhood of 0 in $M(n,\mathbb{R})$ and W_0 a neighbourhood of I in $GL(n,\mathbb{R})$ such that the exponential map is a diffeomorphism from U onto W_0 which maps $U \cap \mathcal{G}$ onto $W_0 \cap G$. The composed map $\Phi = L(g) \circ \exp$ maps U onto $W = gW_0$, and $U \cap \mathcal{G}$ onto $W \cap G$. An important consequence of Theorem is that the set $\exp g$ is a neighbourhood of I in G, hence generates the identity component G_0 of G by Proposition ${}_{k=1}^{\infty}(\exp g)^k = G_0$. ■

Corollaire 1.3.6 *If two closed subgroups G_1 and G_2 of $GL(n,\mathbb{R})$ have the same Lie algebra then the identity components of G_1 and G_2 are the same. It also follows from Theorem that the group G is discrete if and only if its Lie algebra reduces to $\{0\}$: $Lie(G) = \{0\}$. To every closed subgroup G of $GL(n,\mathbb{R})$ one associates its Lie algebra $\mathcal{G} = Lie(G) \subset M(n,\mathbb{R})$. However, not every Lie subalgebra of $M(n,\mathbb{R})$ corresponds to a closed subgroup of $GL(n,\mathbb{R})$.*

1.4 Campbell–Hausdorff formula

Let G be a linear Lie group and $\mathcal{G} = Lie(G)$ its Lie algebra. The Campbell–Hausdorff formula expresses $\log(\exp X \exp Y)$ $(X, Y \in \mathcal{G})$ in terms of a series, each term of which is a homogeneous polynomial in X and Y involving iterated brackets. Let us introduce the functions

$$\Phi(z) = \frac{1-e^{-z}}{z} =_{k=0}^{\infty} (-1)^k \frac{z^k}{(k+1)!}(z \in \mathbb{C})$$

$$\Psi(z) = \frac{z\log z}{z-1} = z_{k=0}^{\infty}(-1)^k \frac{(z-1)^k}{(k+1)}, (|z-1| < 1)$$

If $|z| < \log 2$, then $|e^z - 1| \leq e^{|z|} - 1 < 1$, and $\Psi(e^z)\Phi(z) = \frac{e^z z}{e^z-1}\frac{1-e^{-z}}{z} = 1$. Therefore, if L is an endomorphism such that $\|L\| < \log 2$, then $\Psi(ExpL)\Phi(L) = Id$. With this notation the differential of the exponential map can be written $(D\exp)_A = \exp A\Phi(adA)$.

Théorème 1.4.1 *If* $\|X\|, \|Y\| < r = \frac{1}{2}\log(2 - \frac{1}{2}\sqrt{2})$ *then*

$$\log(\exp X \exp Y) = X + \int_0^1 \Psi(Exp(adX)Exp(tadY))Y\,dt$$

Lemme 1.4.2 *If* $\|X\|, \|Y\| < \alpha$, *then* $\|\exp X \exp Y - I\| \leq e^{2\alpha} - 1$.

Preuve. $\exp X \exp Y - I = (\exp X - I)(\exp Y - I) + (\exp X - I) + (\exp Y - I)$, and, since $\|\exp X - I\| \leq e^{\|X\|} - 1 \leq e^{\alpha} - 1$, $\|\exp X \exp Y - I\| \leq (e^{\alpha} - 1)^2 + 2(e^{\alpha} - 1) = e^{2\alpha} - 1$. ■

Lemme 1.4.3 *If* $\|g - I\| \leq \beta < 1$, *then* $\|\log g\| \leq \log \frac{1}{1-\beta}$.

Preuve. $\|\log g\| \leq_{k=0}^{\infty} \frac{\|(g-I)^k\|}{k} \leq_{k=0}^{\infty} \frac{\beta^k}{k} = \log \frac{1}{1-\beta}$. Let us now prove Theorem. For $\|X\|, \|Y\| < \frac{1}{2}\log 2$, put $F(t) = \log(\exp X \exp tY)$. By Lemma, the function F is defined for $|t| \leq 1$ 1. If furthermore $\|X\|, \|Y\| < r$ (observe that $r < \frac{1}{2}\log 2$, then, by Lemmas $\|F(t)\| < \frac{1}{2}\log 2$ From the inequality $\|XY - YX\| \leq 2\|X\|\|Y\|$ it follows that $\|adX\| \leq 2\|X\|$, hence $\|adF(t)\| < \log 2$. Let us prove that the function F satisfies the differential equation $F'(t) = \Psi(Exp(adF(t))Y$. One can write $\exp F(t) = \exp X \exp tY$. Taking the derivative at t :

$(D\exp p)_{F(t)}(F'(t)) = (\exp X \exp tY)Y$. By Theorem, we obtain $\Phi(adF(t))F'(t) = Y$. Since $\|adF(t)\| < \log 2$ this can be written $F'(t) = \Psi(Exp(adF(t)))Y$. We can also write

$$\begin{aligned} F'(t) &= \Psi(Ad(\exp F(t))Y \\ &= \Psi(Ad(\exp X)Ad(\exp tY))Y \\ &= \Psi(Exp(adX)Exp(adtY))Y. \end{aligned}$$

Furthermore $F(0) = \log(\exp X) = X$, and $F(1) = F(0) + \int_0^1 F'(t)dt$, hence $\log(\exp X \exp Y) = X + \int_0^1 \Psi(Exp(adX)Exp(tadY))Y\,dt$. ■

Théorème 1.4.4 *(Campbell–Hausdorff formula) If $\|X\|, \|Y\| < r = \frac{1}{2}\log(2-\frac{1}{2}\sqrt{2})$, then*

$$\log(\exp X \exp Y) = X +_{k=0}^{\infty} \frac{(-1)^k}{k+1} {}_{\varepsilon(k)} \frac{1}{(q_1+q_2+...q_k+1)} \frac{(adX)^{p_1}(adY)^{q_1}....(adX)^{p_k}(adY)^{q_k}(adX)^m}{p_1!q_1!......p_k!q_k!m!} Y$$

where, for $k \geq 1$, $\varepsilon(k) = \{p_1, q_1, ..., p_k, q_k, m \in N | p_i + q_i > 0, i = 1, ..., k\}$, and $\varepsilon(0) = \{m \in N\}$.

Preuve. If A and B are two endomorphisms

$$(\exp A \exp B - I)^k \exp A =_{\varepsilon(k)} \frac{(A)^{p_1}(B)^{q_1}....(A)^{p_k}(B)^{q_k}(A)^m}{p_1!q_1!......p_k!q_k!m!}$$

Since

$$\Psi(z) = \frac{z\log z}{z-1} = z_{k=0}^{\infty}(-1)^k \frac{(z-1)^k}{(k+1)}, (|z-1| < 1)$$

we have

$$\Psi(Exp(adX)Exp(adtY))Y = Exp(adX)Exp(adtY)Y_{k=0}^{\infty}\frac{(-1)^k}{(k+1)}(Exp(adX)Exp(adtY))-I)^k$$

Observing that $Exp(tadY)Y = Y$, we obtain

$$\Psi(Exp(adX)Exp(adtY))Y =_{k=0}^{\infty} \frac{(-1)^k}{k+1}{}_{\varepsilon(k)} \frac{(adX)^{p_1}(adY)^{q_1}....(adX)^{p_k}(adY)^{q_k}(adX)^m}{p_1!q_1!......p_k!q_k!m!}Y$$

The convergence of the series is uniform for t in [0, 1]. The statement is obtained by termwise integration since

$$\int_0^1 t^{q_1+q_2+...q_k}dt = \frac{1}{(q_1+q_2+...q_k+1)}$$

■

Corollaire 1.4.5

$$\log(\exp X \exp Y) = X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]+\frac{1}{12}[Y,[Y,X]]+termsof\deg ree \geq 4.$$

Chapitre 2

Lie algebras

In this chapter we consider Lie algebras from an algebraic point of view. We willLie algebras In this chapter we consider Lie algebras from an algebraic point of view.We will see how some properties of linear Lie groups can be deduced from the corresponding properties of their Lie algebras. Then we present the basic properties of nilpotent, solvable, and semi-simple Lie algebras.

Définition 2.0.6 *A Lie algebra over $K = \mathbb{R}$ or $\mathbb{C}$ is a vector space $\mathcal{G}$ equipped with a bilinear map $\mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$, $(X, Y) \longmapsto [X, Y]$ satisfying*

(1) $[Y, X] = -[X, Y]$

(2) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$. *The equality* (2) *is called the Jacobi identity. Assume $\mathcal{G}$ is finite dimensional, and let $(X_1, ..., X_n)$ be a basis of $\mathcal{G}$. One can write*

$$[X_i, X_j] =_{k=1}^{n} c_{ij}^k X_k$$

The numbers c_{ij}^k are called the structure constants of the Lie algebra $\mathcal{G}$. Property (1) *can be written $c_{ij}^k = -c_{ji}^k$, and property* (2) *says that, for any m, ${}_{l=1}^{n}(c_{ij}^l c_{lk}^m + c_{jk}^l c_{li}^m + c_{ki}^l c_{lj}^m) = 0$ An automorphism of a Lie algebra is a linear automorphism $y \in GL(g)$ such that*

$$[yX, yY] = y[X, Y].$$

The group of all automorphisms of the Lie algebra $\mathcal{G}$ is denoted by $Aut(\mathcal{G})$. If $\mathcal{G}$

is finite dimensional, it is a closed subgroup of $GL(g)$. A derivation of $\mathcal{G}$ is a linear endomorphism $D \in End(\mathcal{G})$ such that

$$D([X,Y]) = [DX,Y] + [X,DY].$$

For $X \in \mathcal{G}$ let ad X denote the endomorphism of g defined by $adX.Y = [X,Y]$. The Jacobi identity (2) says that ad X is a derivation. The space $Der(\mathcal{G})$ of the

derivations of $\mathcal{G}$ is a Lie algebra for the bracket defined by

$$[D_1, D_2] = D_1 D_2 - D_1 D_2,$$

and the map ad : $g \longrightarrow Der(\mathcal{G})$ is a Lie algebra morphism :

$$ad[X,Y] = [adX, adY].$$

Proposition 2.0.7 *Let $\mathcal{G}$ be a finite dimensional Lie algebra. The Lie algebra of $Aut(\mathcal{G})$ is equal to $Der(\mathcal{G})$.*

Preuve. Let $D \in Lie(Aut(\mathcal{G}))$. For every $t \in \mathbb{R}, Exp(tD)$ is an automorphism of $\mathcal{G}$: for $X, Y \in \mathcal{G}$,

$$Exp(tD)[X,Y] = [Exp(tD)X, Exp(tD)Y]$$

. Taking derivatives of both sides at t = 0 we obtain

$$D[X,Y] = [DX,Y] + [X,DY],$$

which means that D is a derivation : $D \in Der(\mathcal{G})$. Conversely, let $D \in Der(\mathcal{G})$ and put, for $X, Y \in \mathcal{G}$,

$$F_1(t) = Exp(tD)[X,Y],$$

$$F_2(t) = [Exp(tD)X, Exp(tD)Y]$$

We have

$$F_1'(t) = DExp(tD)[X,Y] = DF_1(t),$$

$$F_2'(t) = [DExp(tD)X, Exp(tD)Y] + [Exp(tD)X, DExp(tD)Y],$$

and, since D is a derivation of $\mathcal{G}$,

$$F_2(t) = D\,[Exp(tD)X, Exp(tD)Y] = DF_2(t).$$

Thus F_1 and F_2 are solutions of the same differential equation with the same initial data : $F_1(0) = F_2(0) = [X,Y]$. Hence, for every $t \in \mathbb{R}, F_1(t) = F_2(t)$. This means that, for every t, $Exp(tD)$ is an automorphism of $\mathcal{G}$, and that $D \in Lie(Aut(g))$. An ideal J of a Lie algebra $\mathcal{G}$ is a subalgebra which furthermore satisfies $\forall X \in J, \forall Y \in g, [X,Y] \in J$. Let G be a linear Lie group, and H a closed subgroup. Then $h = Lie(H)$ is a subalgebra of $\mathcal{G} = Lie(G)$ and, if H is a normal subgroup of G, then h is an ideal of $\mathcal{G}$. The converse holds if G and H are connected. Let G be a topological group and V a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. A representation of G on V is a continuous map

$$\pi : G \longrightarrow GL(V),$$

which is a group morphism :

$$\pi(g_1g_2) = \pi(g_1)\pi(g_2)\ (g_1, g_2 \in G),\ \pi(e) = Id,$$

A vector subspace $W \subset V$ is said to be invariant if, for every $g \in G, \pi(g)W = W$. Let us denote by $\pi_0(g)$ the restriction of $\pi(g)$ to $W : \pi_0(g) = \pi(g)W$. Then π_0 is a representation of G onW, one says that π_0 is a subrepresentation of π. The representation π_1 of G on the quotient space V/W is called a quotient representation. The representation π is said to be irreducible if the only invariant subspaces are $\{0\}$ and V. Two representations (π_1, V_1) and (π_2, V_2) are said to be equivalent if there exists an isomorphism $A : V1 \longrightarrow V2$ (A is an invertible linear map) such that

$$A\pi_1(x) = \pi_2(x)A$$

, for every $x \in G$. One says that A is an intertwinning operator or that A intertwins the representations π_1 and π_2.

A representation of a Lie algebra $\mathcal{G}$ on a vector space V is a linear map

$$\rho : \mathcal{G} \longrightarrow End(V)$$

which is a Lie algebra morphism :

$$\rho([X,Y]) = [\rho(X), \rho(Y)] = \rho(X)\rho(Y) - \rho(Y)\rho(X).$$

One also says that V is a module over $\mathcal{G}$, or that V is a $\mathcal{G}$-module.

The map $ad : \mathcal{G} \longrightarrow Der(\mathcal{G}) \subset End(\mathcal{G})$ is a representation of $\mathcal{G}$, which is called the adjoint representation. Let G be a linear Lie group with Lie algebra $\mathcal{G}$ and let π be a representation of G on a finite dimensional vector space V. Then, for $X \in \mathcal{G}$, $t \to \gamma(t) = \pi(\exp tX)$ is a one parameter subgroup of $GL(V)$, hence differentiable by Theorem. Put

$$d\pi(X) = \left(\frac{d}{dt}\pi(\exp tX)\right)_{t=0} (X \in \mathcal{G}),$$

then $d\pi$ is a representation of the Lie algebra of $\mathcal{G}$ on V, which is called the derived representation of π. Let us prove this fact. By Theorem,

$$\pi(\exp X) = Expd\pi(X)\ (X \in g).$$

From the definition of $d\pi$ it follows at once that, for $t \in \mathbb{R}, d\pi(tX) = td\pi(X)$. By Corollary

$$\begin{aligned}
\pi(\exp t(X+Y)) &= \lim_{k\to\infty}(\pi(\exp\frac{tX}{k})\pi(\exp\frac{tY}{k}))^k \\
&= \lim_{k\to\infty}(Exp\frac{d\pi(tX)}{k}Exp\frac{d\pi tY}{k})^k \\
&= Exp(d\pi(tX) + d\pi(tY)) \\
&= Exp(td\pi(X) + td\pi(Y))
\end{aligned}$$

and, by taking the derivatives at $t = 0$, we get

$$d\pi(X+Y) = d\pi(X) + d\pi(Y).$$

Furthermore

$$\pi\left(\exp(tAd(x)Y)\right) = \pi(x)\pi(\exp tY)\pi(x^{-1}).$$

By taking the derivatives at $t = 0$, we get

$$d\pi(Ad(g)Y) = \pi(x)d\pi(Y)\pi(x^{-1}).$$

Then put $x = \exp sX$ and take the derivatives at $s = 0$,

$$d\pi([X, Y]) = d\pi(X)d\pi(Y) - d\pi(Y)d\pi(X).$$

The adjoint representation $\pi = Ad$ of G on $\mathcal{G}$ is a special case for which the derived representation is the adjoint representation ad of $\mathcal{G}$ on $\mathcal{G}$. If π_1 and π_2 are two equivalent representations, then the derived representations $d\pi_1$ and $d\pi_2$ are also equivalent. The converse holds if G is connected. The kernel of a representation of a Lie algebra is an ideal. The center of a Lie algebra $\mathcal{G}$, denoted by $Z(\mathcal{G})$, is defined as

$$Z(\mathcal{G}) = \{X \in \mathcal{G} | \forall Y \in \mathcal{G},\ [X, Y] = 0\}.$$

It is an Abelian ideal. It is the kernel of the adjoint representation. ■

Remarque 2.0.8 *One can show that every finite dimensional Lie algebra admits a faithful (i.e. injective) finite dimensional representation. This is the theorem of Ado. Hence every finite dimensional Lie algebra can be seen as a subalgebra of $gl(N, \mathbb{k}) = M(N, \mathbb{k})$, for some N.*

Let G and H be two linear Lie groups and φ a continuous morphism of G into H. One puts, for $X \in \mathcal{G} = Lie(G)$,

$$d\varphi(X) = \frac{d}{dt}\varphi(\exp tX) \restriction_{t=0} .$$

From what we have seen, $d\varphi$ is a Lie algebra morphism from $\mathcal{G}$ into $h = Lie(H)$. Observe that $d\varphi$ is the differential of φ at the identity element I of G :

$$d\varphi = (D\varphi)I.$$

Proposition 2.0.9 (i) *The Lie algebra of the kernel of the morphism φ is equal to the kernel of $d\varphi$:*

$$Lie(\ker \varphi) = \ker(d\varphi).$$

Therefore the kernel of φ is discrete if and only if $d\varphi$ is injective.

(ii) *If $d\varphi$ is surjective, then the image of φ contains the identity component H_0 of H.*

(iii) *If G and H are connected and if $d\varphi$ is an isomorphism, then (G, φ) is a covering of H.*

Let us recall the definition of a covering. Let X and Y be two connected topological spaces and $\varphi : X \longrightarrow Y$ a continuous map. The pair (X, φ) is called a covering of Y if φ is surjective and if, for every $x \in X$, there exist neighbourhoods V of x and W of $y = \varphi(x)$ such that the restriction of φ to V is a homeomorphism from V onto W. Let (X, φ) be a covering of Y ; if, for $y_0 \in Y$, the pullback $\varphi^{-1}(y_0) \subset X$ is a finite set, then the same holds for every $y \in Y$, and the pullbacks $\varphi^{-1}(y)$ all have the same number of elements. Let k be that number. Then one says that (X, φ) is a covering of order k of Y (or a covering with k sheets).

Preuve. (a) From Theorem it follows that, for $X \in \mathcal{G}$

$, t \in \mathbb{R}$,

$$\varphi(\exp tX) = \exp(td\varphi(X))$$

Hence

$$Lie(\ker \varphi) = \ker(d\varphi).$$

In particular, $d\varphi$ is injective if and only if the Lie algebra of ker φ reduces to $\{0\}$, that is if $\ker \varphi$ is discrete.

(b) Recall that G_0 denotes the identity component of G. Assume that $d\varphi$ is surjective. This means that the differential of φ at the identity element e of G is surjective. Then $V = \varphi(G_0)$ is a neighbourhood of the identity element e of H. We saw that

$$H_0 =_{k=1}^{\infty} V^k$$

(Proposition). Since φ is a group morphism, $V = \varphi(G_0)$ is a subgroup of H, and $V^k = V$, hence $H_0 = \varphi(G_0)$.

(c) Assume that G and H are connected and that $d\varphi$ is an isomorphism. Let us show that (G, φ) is a covering of H. From (ii) it follows that φ is surjective. By using Theorem and the relation

$$\varphi(\exp X) = \exp(d\varphi(X)) \ (X \in g),$$

one can show that there is a neighbourhood $V \subset G$ of the identity element of G, and a neighbourhood $W \subset H$ of the identity element of H such that φ is an isomorphism from V onto W. It follows that, for every $g \in G$, φ is a homeomorphism of the neighbourhood gV of g onto the neighbourhood hW of $h = \varphi(g)$ since

$$\varphi(gv) = h\varphi(v) \ (v \in V).$$

If $\ker \varphi$ is a finite group, then (G, φ) is a covering of order k of H, where k is the number of elements in $\ker \varphi$. ∎

Exemple 2.0.10 *Let V be the vector space of 2×2 Hermitian matrices with zero trace. Such a matrix can be written* $x = \begin{pmatrix} x_1 & x_2 + ix_3 \\ x_2 - ix_3 & -x_1 \end{pmatrix}$ $(x_1, x_2, x_3 \in \mathbb{R})$.

Then $V \approx \mathbb{R}^3$. For $g \in G = SU(2)$ the transformation

$$x \longrightarrow gxg^{-1} = gxg^*,$$

is a linear map $\pi(g)$ from V onto V. From the relation

$$\det x = -x_1^2 - x_2^2 - x_3^2,$$

it follows that the transformation $\pi(g)$ is orthogonal. Then one gets a morphism φ from $SU(2)$ into $O(3)$. For $T \in su(2), d\pi(T)x = Tx - xT$. If,

$$T = \begin{pmatrix} iu & v + iw \\ -v + iw & -iv \end{pmatrix}$$

one can establish easily that the matrix of $d\pi(T)$ *is*

$$\begin{pmatrix} 0 & 2v & 2w \\ -2v & 0 & -2u \\ -2w & 2u & 0 \end{pmatrix}$$

Therefore $d\varphi$ *is a bijection from* $su(2)$ *onto* $LieO(3) = Asym(3, \mathbb{R})$*. The group* $SU(2)$ *is connected. It follows that the group* $\varphi(G)$ *is the identity component of* $O(3)$*, that is* $SO(3)$*. The kernel of* φ *is discrete. In fact one can check that* $\ker \varphi = \{I, -I\}$*. This establishes that* $SO(3) \simeq SU(2)/\{\pm I\}$*, and that* $(SU(2), \varphi)$ *is a covering of order two of* $SO(3)$*.*

2.1 Nilpotent and solvable Lie algebras

Let us recall some definitions and notation in group theory. Let G be a group. If $\{e\}$ and G are the only normal subgroups, G is said to be simple. If G is commutative, every subgroup is normal. The commutator of two elements x and y of G is defined as

$$[x, y] = x^{-1}y^{-1}xy.$$

The derived group $D(G)$ is the subgroup of G which is generated by the commutators. If H is a normal subgroup, then G/H is a group. It is commutative if and only if H contains the derived group $D(G)$. One defines the successive derived groups : $D_0(G) = G$ and $D_{i+1}(G) = D(D_i(G))$. The group G is said to be solvable if there exists an integer $n \geq 0$ such that $D_n(G) = \{e\}$. (The terminology comes from the fact that, in Galois theory, such groups make it possible to characterise polynomial equations which are solvable by radicals.) Let g be a finite dimensional Lie algebra over $\Bbbk = \mathbb{R}$ or $\mathbb{C}$. If A and B are two vector subspaces of g, then $[A, B]$ denotes the vector subspace of g generated by the brackets$[X, Y]$ with $X \in A$ and $Y \in B$. One puts $D(g) = [g, g]$. This is an ideal of g which is called the derived ideal. The descending central series $C^k(\mathcal{G})$ is defined recursively by :

$$C^1(\mathcal{G}) = \mathcal{G}, C^k(\mathcal{G}) = [C^{k-1}(\mathcal{G}), \mathcal{G}].$$

It is also denoted by $C^k(\mathcal{G}) = \mathcal{G}^k$. Observe that $C^2(\mathcal{G}) = D(\mathcal{G})$. The derived series is defined by

$$D^1(\mathcal{G}) = D(\mathcal{G}),\ D^k(\mathcal{G}) = D(D^{k-1}(\mathcal{G})) = \left[D^{k-1}(\mathcal{G}), D^{k-1}(\mathcal{G})\right]$$

It is also denoted by $D^k(\mathcal{G}) = \mathcal{G}^{(k)}$. The subspaces $C^k(g)$ and $D^k(\mathcal{G})$ $(k = 1, 2, ...)$ are ideals. The sequence $C^k(\mathcal{G})$ is decreasing, hence constant for k large enough. The Lie algebra g is said to be nilpotent if there exists an integer $n \geq 1$ such that $C^n(\mathcal{G}) = \{0\}$. Similarly the sequence $D^k(\mathcal{G})$ is decreasing, hence constant for k large enough. The Lie algebra $\mathcal{G}$ is said to be solvable if there exists $n \geq 1$ such that $D^n(\mathcal{G}) = \{0\}$. Observe that a nilpotent Lie algebra is solvable. A subalgebra of a nilpotent Lie algebra is nilpotent. A subalgebra of a solvable Lie algebra is solvable. Let X be an element in a nilpotent Lie algebra $\mathcal{G}$, then adX is a nilpotent endomorphism. (Recall that an endomorphism T is said to be nilpotent if there exists an integer $k \geq 1$ such that $T^k = 0$.) In fact adX maps $C^k(\mathcal{G})$ into $C^{k+1}(\mathcal{G})$ and, if $C^n(\mathcal{G}) = \{0\}$, then $(adX)^{n-1} = 0$.

Exemple 2.1.1 (1) *Let G be the group '$ax + b$', that is the group of affine transformations of $\mathbb{R}$. The Lie algebra $\mathcal{G} = Lie(G)$ has dimension 2. It has a basis $\{X_1, X_2\}$ satisfying*

$$[X_1, X_2] = X_1.$$

Hence $D(\mathcal{G}) = \mathbb{R}X_1$, $C^3(\mathcal{G}) = C^2(\mathcal{G}) = \mathbb{R}X_1$, $D^2(\mathcal{G}) = \{0\}$. Therefore $\mathcal{G}$ is solvable, but not nilpotent.

(2) *The Heisenberg Lie algebra $\mathcal{G}$ of dimension 3 has a basis $\{X_1, X_2, X_3\}$ satisfying*

$$[X_1, X_2] = X_3,\ [X_1, X_3] = 0,\ [X_2, X_3] = 0.$$

Hence $C^2(\mathcal{G}) = \Bbbk X_3$, which is the centre of $\mathcal{G}$, and $C^3(\mathcal{G}) = \{0\}$. Therefore $\mathcal{G}$ is nilpotent.

(3) *Let* $\mathcal{G} = sl(2, \Bbbk)$ *be the Lie algebra of the group* $SL(2, \Bbbk)$*. It has a basis* $\{X_1, X_2, X_3\}$ *satisfying*

$$[X_1, X_2] = 2X_2, \ [X_1, X_3] = -2X_3, \ [X_2, X_3] = X_1$$

. Hence $D(\mathcal{G}) = \mathcal{G}$*. Therefore* $\mathcal{G}$ *is neither nilpotent, nor solvable.*

(4) *Let* $T_0(n, \Bbbk)$ *be the group of upper triangular matrices with diagonal entries equal to one. Its Lie algebra* $g = t_0(n, \Bbbk)$ *consists of the upper triangular matrices with zero diagonal entries*

$$t_0(n, \Bbbk) = \{x \in M(n, \Bbbk) | \ x_{ij} = 0 \ \textit{if} \ i \geq j\}.$$

For $1 \leq k \leq n-1$, $C^k(\mathcal{G}) = \{x \in g | \ x_{ij} = 0 \textit{if} \ i \geq j - k + 1\}$*. In particular* $C_n(\mathcal{G}) = \{0\}$*, and* g *is nilpotent. This is the basic example of a nilpotent Lie algebra.*

(5) *Let* $T(n, \Bbbk)$ *be the group of upper triangular matrices with non-zero diagonal entries. Its Lie algebra* $g = t(n, \Bbbk)$ *consists of the upper triangular matrices,*

$$t(n, \Bbbk) = \{x \in M(n, \Bbbk) | x_{ij} = 0 \ \textit{if} \ i > j\}.$$

We have

$$C^2(\mathcal{G}) = C^3(\mathcal{G}) = ... = t_0(n, \Bbbk),$$

$$D^k(\mathcal{G}) = \{x \in M(n, \Bbbk) | x_{ij} = 0 \ \textit{if} \ i > j - 2^{k-1}\}.$$

Hence

$$D^k(g) = \{0\} \ \textit{if} \ 2^{k-1} \geq n - 1.$$

Therefore g *is solvable, but is not nilpotent. This is the basic example of a solvable Lie algebra. Let* g *be a Lie algebra and* ρ *a representation of* g *on a finite dimensional vector space* V*. The representation* ρ *is said to be nilpotent if, for every* X *of* g*, the endomorphism* $\rho(X)$ *is nilpotent.*

Lemme 2.1.2 *If* X *is a nilpotent endomorphism acting on a vector space* V*, then* adX *is nilpotent.*

Preuve. Let $k \geq 1$ be such that $X_k = 0$. We have

$$(adX)^N = (L_X - R_X)^N =_{j=0}^{N} (-1)^{N-j} C_j^N L_{X^j} R_{X^{N-j}}.$$

Hence, if $N \geq 2k - 1$, then $(adX)^N = 0$. ■

Théorème 2.1.3 *Let ρ be a nilpotent representation of a Lie algebra $\mathcal{G}$ on a vector space V. There exists a vector $v_0 = 0$ in V such that, for every $X \in \mathcal{G}$, $\rho(X)v_0 = 0$.*

Preuve. Let $\ker(\rho)$ be the kernel of ρ. It is an ideal of $\mathcal{G}$. It is enough to prove the statement for the representation ρ of the quotient algebra $\mathcal{G}/\ker(\rho)$. This representation is faithful (i.e. injective). Hence we may assume that $\mathcal{G}$ is a subalgebra of $gl(V)$. We have to show the following statement : if $\mathcal{G}$ is a Lie subalgebra of $gl(V)$ made of nilpotent endomorphisms, then there exists $v_0 = 0$ in V such that, for every $X \in \mathcal{G}, Xv_0 = 0$. The statement will be proved recursively with respect to the dimension of $\mathcal{G}$. If $\dim \mathcal{G} = 1$, then $\mathcal{G} = \Bbbk X$, and X is nilpotent. Hence there exists $v_0 = 0$ in V such that $Xv_0 = 0$. Assume that the property holds for every Lie algebra with dimension $\leq n - 1$.

(a) Let $\mathcal{G}$ be a subalgebra of dimension n of $gl(V)$ made of nilpotent endomorphisms, and let h be a proper subalgebra of $\mathcal{G}$ with maximal dimension.We will show that h is an ideal of dimension $n - 1$. Let us consider the representation α of h on $W = \mathcal{G}/h$ defined by

$$\alpha(X) : Y + h \longrightarrow [X, Y] + h.$$

By Lemma it follows that the representation α is nilpotent.Bythe recursion assumption it follows that there exists $w_0 = 0$ in W such that, for every X in h,

$$\alpha(X)w_0 = 0.$$

Let $X_0 \in \mathcal{G}$ be a representative of w_0. Then X_0 does not belong to h and $[X_0, h] \subset h$. Hence $\Bbbk X_0 + h$ is a subalgebra of g whose dimension is greater than that of h, therefore $\mathcal{G} = \Bbbk X_0 + h$, and $\dim h = n - 1$. Furthermore $[\mathcal{G}, h] \subset h$, and this means that h is an ideal.

(b) Let us use for a second time the recursion assumption : there exists $v_1 = 0$ in V such that, for every X in h,

$$Xv_1 = 0.$$

Put

$$V_0 = \{v \in V | \forall X \in h, Xv = 0\}.$$

Since $v_1 \in V_0$, $V_0 = \{0\}$. The subspace V_0 is invariant under $\mathcal{G}$. In fact let $X \in \mathcal{G}, v \in V_0$, and show that $Xv \in V_0$. For

$$Y \in h, YXv = XYv - [X, Y]v = 0.$$

In particular, $X_0 V_0 \subset V_0$. Since X_0 is nilpotent there exists in V_0 a vector $v_0 = 0$ such that $X_0 v_0 = 0$, and then, for every X in $\mathcal{G}$, $Xv_0 = 0$. ■

Théorème 2.1.4 *Let ρ be a nilpotent representation of a Lie algebra $\mathcal{G}$ on a vector space V. There exists a basis of V such that, for every X in $\mathcal{G}$, the matrix of $\rho(X)$ is upper triangular with zero diagonal entries.*

Preuve. Let us prove the statement recursively with respect to the dimension of V. By Theorem there exists a vector v_1 such that, for every $X \in g$,

$$\rho(X)v_1 = 0.$$

From the recursion assumption applied to the quotient $W = V/Kv_1$ we get the result. ■

Corollaire 2.1.5 *(Engel's Theorem) A Lie algebra is nilpotent if and only if, for every $X \in \mathcal{G}$, adX is a nilpotent endomorphism*

Preuve. (a) Assume that the Lie algebra g is nilpotent : there exists an integer n such that $C^n(\mathcal{G}) = \{0\}$. For every X in $\mathcal{G}$, adX maps $C^k(\mathcal{G})$ into $C^{k+1}(\mathcal{G})$, hence $(adX)^{n-1} = 0$. (b) Assume that, for every X in $\mathcal{G}$, adX is nilpotent. By Theorem the Lie algebra $ad\mathcal{G}$ is isomorphic to a subalgebra of $t_0(N, \mathbb{k})$, hence ad $\mathcal{G}$ is nilpotent. There exists an integer n such that $C^n(ad\mathcal{G}) = \{0\}$, hence $C^n(\mathcal{G}) \subset Z(\mathcal{G})$, the centre of $\mathcal{G}$. Therefore $C^{n+1}(\mathcal{G}) = \{0\}$. Let I be an ideal of $\mathcal{G}$. If g is solvable, then g/I is solvable too. In fact,

$$D^k(g/I) \simeq D^k(\mathcal{G})/I \cap D^k(\mathcal{G})$$

■

Proposition 2.1.6 *If I and $\mathcal{G}/I$ are solvable, then $\mathcal{G}$ is solvable.*

Preuve. There is an integerm such that $D^m(\mathcal{G}/I) = \{0\}$, hence$D^m(g) \subset I$. There exists n such that $D^n(I) = \{0\}$. Therefore $D^{m+n}(\mathcal{G}) = \{0\}$. ■

Proposition 2.1.7 *If I_1 and I_2 are two solvable ideals then the ideal $I_1 + I_2$ is also solvable.*

Preuve. The Lie algebra $(I_1 + I_2)/I_2$ is isomorphic to $I_1/I_1 \cap I_2$. This follows from the preceding proposition. Hence, if $\mathcal{G}$ is finite dimensional, there exists a largest solvable ideal : the sum of all solvable ideals. It is called the radical of $\mathcal{G}$, and is denoted by $rad(\mathcal{G})$. ■

Théorème 2.1.8 *(Lie's Theorem) Let $\mathcal{G}$ be a solvable Lie algebra over $\mathbb{C}$, and let ρ be a representation of $\mathcal{G}$ on a finite dimensional complex vector space V. There exists a vector $v_0 = 0$ in V, and a linear form λ on $\mathcal{G}$ such that, for every X in $\mathcal{G}$, $\rho(X)v_0 = \lambda(X)v_0$.*

Preuve. We will prove the statement recursively with respect to the dimension of $\mathcal{G}$. If $\dim \mathcal{G} = 1$, then $g = \mathbb{C}X_0$, and $\rho(X_0)$ has an eigenvector. Assume that the property holds for every solvable Lie algebra of dimension $\leq n-1$. Let g be a solvable Lie algebra of dimension n, and let h be a subspace of $\mathcal{G}$ of dimension $n-1$ containing $D(\mathcal{G})$. Such a subspace exists since, because $\mathcal{G}$ is solvable, $D(\mathcal{G}) = \mathcal{G}$. The subspace h is an ideal because

$$[\mathcal{G}, h] \subset [\mathcal{G}, \mathcal{G}] = D(\mathcal{G}) \subset h.$$

By the recursion assumption there is a vector $w_0 = 0$ in V and a linear form λ on h such that, for every Y in h,

$$\rho(Y)w_0 = \lambda(Y)w_0.$$

Let $X_0 \in g\backslash h$, and put

$$wj = \rho(X0)jw0, j \geq 1.$$

Let k be the largest integer for which the vectors $w_0, ..., w_k$ are linearly independent, and let W_j be the subspace which is generated by $w_0, ..., w_j$ $(0 \leq j \leq k)$.Observe that $w_j \in W_k$ for $j \geq k$. Hence $\rho(X_0)$ maps W_k into Wk and, for $0 \leq j < k$, W_j into W_{j+1}. We will show that, for $Y \in h$, the restriction of $\rho(Y)$ to Wk is equal to $\lambda(Y)I$. In a first step we will show that the matrix of $\rho(Y)$ with respect to the basis $\{w_0, ..., w_k\}$ is upper triangular with diagonal entries equal to $\lambda(Y)$. Let us show recursively with respect to $j(0 \leq j \leq k)$ that

$$\rho(Y)w_j = \lambda(Y)w_j$$

mod W_{j-1}. (One puts $W_{-1} = \{0\}$.) This holds clearly for$j = 0$. Assume that it holds for $j < k$. Then, for $Y \in h$,

$$\rho(Y)w_{j+1} = \rho(Y)\rho(X_0)w_j = \rho(X_0)\rho(Y)w_j + \rho([Y, X_0])w_j,$$

and, since $[Y, X_0] \in h$,

$$\rho([Y, X_0])w_j = \lambda([Y, X_0])w_j$$

mod W_{j-1} by the recursion assumption. Hence

$$\rho(Y)w_{j+1} = \lambda(Y)w_{j+1}$$

mod W_j. This shows that the subspace W_k is invariant under the representation ρ. For $Y \in h$,

$$Tr(\rho([Y, X_0])W_k) = 0.$$

On the other hand, for $Z \in h$,

$$Tr(\rho(Z) \upharpoonright_{W_k}) = (k + 1)\lambda(Z).$$

Hence, if $Z = [Y, X_0]$, then $\lambda(Z) = 0$. In a second step we will showthat, for $Y \in h$, and $w \in W_k$, $\rho(Y)w = \lambda(Y)w$. Let us show recursively with respect to $j(0 \leq j \leq k)$ that, for $Y \in h$,

$$\rho(X)w_j = \lambda(Y)w_j.$$

This holds for $j = 0$. Assume that $\rho(Y)w_j = \lambda(Y)w_j$. Then

$$\begin{aligned} \rho(Y)w_{j+1} &= \rho(X_0)\rho(Y)w_j + \rho([Y, X_0])w_j \\ &= \lambda(Y)\rho(X_0)w_j + \lambda([Y, X_0])w_j = \lambda(Y)w_{j+1}. \end{aligned}$$

Let $v_0 \in W_k$ be an eigenvector of $\rho(X_0)$,

$$\rho(X_0)v_0 = \mu v_0,$$

and extend the linear form λ to g by putting

$$\lambda(X_0) = \mu.$$

Then, for every X in g,

$$\rho(X)v_0 = \lambda(X)v_0.$$

■

Corollaire 2.1.9 *Let $\mathcal{G}$ be a solvable Lie algebra over $\mathbb{C}$, and ρ be a representation of $\mathcal{G}$ on a finite dimensional complex vector space V. There exists a basis of V such that, for every X in $\mathcal{G}$, the matrix of $\rho(X)$ is upper triangular. The diagonal entries can be written $\lambda_1(X), ..., \lambda_m(X)$, where $\lambda_1, ..., \lambda_m$, are linear forms on $\mathcal{G}$.*

The statements of Theorem and Corollary do not hold if $\mathcal{G}$ is a solvable Lie algebra over $\mathbb{R}$. (In fact one knows that, if A is an endomorphism of a finite dimensional real vector space, in general there is no basis with respect to which the matrix of A is upper triangular.) See Exercise.

2.2 Semi-simple Lie algebras.

A Lie algebra is said to be simple if it has no non-trivial ideal, and if it is not commutative. In other words a Lie algebra is simple if its dimension is greater than 1 and if the adjoint representation ad is irreducible. If $\mathcal{G}$ is simple then $[\mathcal{G}, \mathcal{G}] = \mathcal{G}$, because $[\mathcal{G}, \mathcal{G}]$ is an ideal. The Lie algebra $sl(n, \mathbb{k})$ is simple ($n \geq 2$). Let us show that $sl(2, \mathbb{C})$ is simple. (For $n \geq 3$, see Exercise .) For that consider the following basis of $sl(2, \mathbb{C})$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The commutation relations are :

$$[H, E] = 2E, [H, F] = -2F, [E, F] = H.$$

Let I be an ideal of $sl(2, \mathbb{C})$ which does not reduce to $\{0\}$. If one of the elements H, E, or F belongs to J, then $J = sl(2, \mathbb{C})$. The basis elements H, E, F are eigenvectors of ad H for the eigenvalues $0, 2, -2$, and J is invariant under ad H, hence one of the eigenvectors belongs to J. A Lie algebra $\mathcal{G}$ is said to be semi-simple if the only commutative ideal is $\{0\}$. A simple Lie algebra is semi-simple. There is no semi-simple Lie agebra of dimension 1 or 2. But there exist semi-simple Lie algebras of dimension 3 ; in fact $sl(2, \mathbb{C})$, $sl(2, \mathbb{R})$ and $su(2)$ are semi-simple Lie algebras. The centre of a semi-simple Lie algebra reduces to $\{0\}$. Hence, if $\mathcal{G}$ is semisimple, then the adjoint representation is faithful, $ad(\mathcal{G}) \simeq \mathcal{G}$. A direct sum of semi-simple Lie algebras is semi-simple. Let ρ be a representation of a Lie algebra $\mathcal{G}$ on a finite dimensional vector space V. For $X, Y \in \mathcal{G}$ one puts

$$B_\rho(X, Y) = Tr(\rho(X)\rho(Y))$$

. This is a symmetric bilinear form on $\mathcal{G}$ which is associative :

$$B_\rho([X, Y], Z) = B_\rho(X, [Y, Z]).$$

This means that the transformations adX are skewsymmetric with respect to the form B_ρ. The orthogonal of an ideal with respect to the form B_ρ is an ideal. The Killing form is the symmetric bilinear form associated to the adjoint representation ($\rho = ad$) :

$$B(X, Y) = Tr(adXadY).$$

Exemple 2.2.1 *(1) Let $\mathcal{G} = M(n, \mathbb{k})$.*

$$B(X, Y) = 2nTr(XY) - 2TrXTrY.$$

In order to establish this formula let us consider the canonical basis $\{E_{ij}\}$ of $\mathcal{G} = M(n, \mathbb{k})$. If $X =_{i,j=1}^{n} x_{ij}E_{ij} \in \mathcal{G}$, then $ad.E_{kl} = X, E_{k,l}] =_{i,j=1}^{n}$ $(x_{ik}E_{il} - x_{li}E_{ki})$ $(k, = 1, ..., n)$. Hence, if $Y =_{i,j=1}^{n} y_{ij}E_{ij}$ then

$$(adX \circ adY)E_{kl} =_{i,j=1}^{n} (x_{ik}y_{ji}E_{ij} + x_{li}y_{ij}E_{kl}) -_{i,j=1}^{n} (x_{ik}y_{lj} + x_{lj}y_{ij})E_{kl})$$

Therefore

$$\begin{aligned} Tr(adXadY) &= n_{i,j=1}^{n}(x_{ij}y_{ij} + x_{ji}y_{ij}) - 2_{i,j=1}^{n}(x_{ii}y_{jj}) \\ &= 2nTr(XY) - 2TrXTrY. \end{aligned}$$

(2) *Let $g = sl(n, \mathbb{k})$. If $n \geq 2$, $B(X, Y) = 2nTr(XY)$. (This follows from Proposition 4.3.1 below.)*

(3) *Let $g = so(n, \mathbb{k})$. If $n \geq 2$,*

$$B(X, Y) = (n - 2)Tr(XY).$$

The proof is left as an exercise.

Proposition 2.2.2 *Let J be an ideal in a Lie algebra $\mathcal{G}$. The Killing form of the Lie algebra J is the restriction to J of the Killing form of $\mathcal{G}$.*

Preuve.

Théorème 2.2.3 *Let $X, Y \in J$. The endomorphisms $S = adX$, $T = adY$ map $\mathcal{G}$ into J. Let us consider a basis of $\mathcal{G}$ obtained by completing a basis of J. With respect to this basis the matrices of S and T have the following shape*

$Mat(S) = \begin{pmatrix} S_1 & * \\ 0 & 0 \end{pmatrix}$, $Mat(T) = \begin{pmatrix} T_1 & * \\ 0 & 0 \end{pmatrix}$ *and* $Mat(ST) = \begin{pmatrix} S_1T_1 & * \\ 0 & 0 \end{pmatrix}$
Therefore $Tr(ST) = Tr(S_1T_1)$. *We will see that a Lie algebra is semi-simple if and only if the Killing form is non-degenerate. To prove this we will use Cartan's criterion for solvable Lie algebras. We will need the properties of the decomposition of an endomorphism into semi-simple and nilpotent parts. Let* V *be a finite dimensional vector space over* C. *Recall that an endomorphism* T *of* V *decomposes as*

$$T = T_s + T_n,$$

where T_s *is semi-simple (i.e. diagonalisable), and* T_n *is nilpotent, in such a way that* T_s *and* T_n *are polynomials in* T . *The endomorphisms* T_s *and* T_n *commute. This decomposition is unique in the following sense : if*

$$T = D + N,$$

with D *semi-simple,* N *nilpotent, and* $DN = ND$, *then* $D = T_s$, $N = T_n$. T_s *is called the semi-simple part of* T, *and* T_n *the nilpotent part We have*

$$adT = adT_s + adT_n,$$

adT_s *is semi-simple,* adT_n *is nilpotent (Lemma). In order to show that* adT_s *and* adT_n *are the semi-simple and nilpotent parts of* $ad\,T$ *it is enough to show that* adT_s *and* adT_n *commute. But*

$$[ad(T_s), ad(T_n)] = ad[T_s, T_n] = 0.$$

It follows that $ad\ T_s$ *and* $ad\ T_n$ *are polynomials in* $ad\ T$.

■

Théorème 2.2.4 *(Cartan's criterion) Let* $\mathcal{G}$ *be a Lie subalgebra of* $M(m, \mathbb{C})$. *Assume that* $Tr(XY) = 0$ *for every* $X, Y \in \mathcal{G}$. *Then* $\mathcal{G}$ *is solvable.*

Preuve. We will show that every $X \in [\mathcal{G}, \mathcal{G}]$ is a nilpotent endomorphism.

(a) Let $X = X_s + X_n$ be the decomposition of $X \in \mathcal{G}$ into semi-simple and nilpotent parts. We may assume that

$$X_s = \begin{pmatrix} \lambda_1 & & \\ & . & \\ & & \lambda_m \end{pmatrix},$$

the numbers λ_j being the eigenvalues of X. Let p be a polynomial in one variable, and put

$$U = p(X_s) = \begin{pmatrix} p(\lambda_1) & & \\ & . & \\ & & p(\lambda_m) \end{pmatrix},$$

Since X_s is a polynomial in X, one can write $U = p_0(X)$, and since X_n is also a polynomial in X, U and X_n commute. Then

$$(UX_n)^k = U^k X_n^k$$

hence UXn is nilpotent, therefore $Tr(UX_n) = 0$, or $Tr(UX_s) = Tr(UX)$.

(b) The eigenvalues of ad X_s are the numbers $\lambda_i - \lambda_j$, and the corresponding eigenvectors are the matrices E_{ij},

$$adX_s E_{ij} = (\lambda_i - \lambda_j)E_{ij}.$$

Let us now choose a polynomial p in one variable with complex coefficients such that $p(\lambda_i) = \overline{\lambda_i}$ $(i = 1, ..., m)$. Hence, if $\lambda_i - \lambda_j = \lambda_k - \lambda_l$, then

$$p(\lambda_i) - p(\lambda_j) = p(\lambda_k) - p(\lambda_l).$$

Therefore there exists a polynomial P such that, if $U = p(X_s)$, then $adU = P(adX_s)$ and, since adX_s is a polynomial in adX, there exists a polynomial P_0 such that $adU = P_0(adX)$. Therefore $adU(\mathcal{G}) \subset \mathcal{G}$.

(c) Let us now take $X \in [g, g]$, and show that

$$Tr(UX) = 0.$$

We can write $X =_{,j=1}^{n} [Y_j, Z_j]$, $Y_j, Z_j \in \mathcal{G}$, and then

$$Tr(UX) =_{,j=1}^{n} Tr(U [Y_j, Z_j]) =_{,j=1}^{n} Tr([U, Y_j] Z_j) = 0$$

by assumption, since $[U, Y_j] = adUY_j \in \mathcal{G}$. But, by (a)

$$Tr(UX) = Tr(UX_s),$$

hence

$$Tr(UX) =_{,j=1}^{n} p(\lambda_j)\lambda_j =_{,j=1}^{n} |\lambda_j|^2 .$$

Therefore the eigenvalues λ_j of X vanish, and X is nilpotent. By Engel's Theorem (Corollary) it follows that $[\mathcal{G}, \mathcal{G}]$ is nilpotent, hence $\mathcal{G}$ is solvable ■

Corollaire 2.2.5 *If the Killing form of $\mathcal{G}$ vanishes identically, then $\mathcal{G}$ is solvable.*

Théorème 2.2.6 *Let $\mathcal{G}$ be a Lie algebra. The following properties are equivalent :*

(i) $\mathcal{G}$ is semi-simple,

(ii) the radical of g reduces to $\{0\}$,

(iii) the Killing form of g is non-degenerate.

Preuve. (i) $\Rightarrow$ (ii). Assume that there exists a solvable ideal $I \neq \{0\}$ in g. Let $D^{k-1}(I)$ be the last non-zero derived ideal of I. Then $D^{k-1}(I)$ is a non-zero commutative ideal in $\mathcal{G}$, and this contradicts (i).

(ii)$\Rightarrow$(iii). Put $I = \mathcal{G}^{\perp}$,

$$I = \{X \in \mathcal{G} | \forall Y \in \mathcal{G},\ B(X, Y) = 0\}$$

This is an ideal and the restriction of B to I vanishes identically. By Corollary, I is a solvable ideal, and $I = \{0\}$ by (ii).

(iii) $\Rightarrow$ (i). Let I be a commutative ideal in $\mathcal{G}$. For $X \in I, Y \in \mathcal{G}$, the endomorphism $adXadY$ maps g into I, and $(adXadY)^2$ maps g into $[I, I] = \{0\}$, hence $adXadY$ is nilpotent. Therefore

$$B(X, Y) = Tr(adXadY) = 0.$$

Since B is non-degenerate it follows that $I = \{0\}$. ■

Proposition 2.2.7 *A semi-simple Lie algebra $\mathcal{G}$ is a direct sum of simple subalgebras. Furthermore, $[\mathcal{G},\mathcal{G}] = \mathcal{G}$.*

Preuve. Let I be an ideal of $\mathcal{G}$, and let $I^{\perp}$ be its orthogonal complement with respect to the Killing form,

$$I^{\perp} = \{X \in \mathcal{G} | \forall Y \in I, B(X,Y) = 0\}.$$

Since the Killing form is associative it follows that $I^{\perp}$ is an ideal and, by Corollary, that the ideal $I \cap I^{\perp}$ is solvable, hence reduces to $\{0\}$ since the radical of $\mathcal{G}$ reduces to $\{0\}$. Therefore $\mathcal{G} = I \oplus I^{\perp}$. To get the stated decomposition one starts from a minimal non-zero ideal I_1 in g, which is necessarily simple, then one obtains recursively a decomposition

$$\mathcal{G} = I_1 \oplus I_2 \oplus \cdots \oplus I_m,$$

where $I_1, ..., I_m$ are simple ideals. It follows furthermore that

$$[\mathcal{G},\mathcal{G}] =_{i=1}^{m} [I_i, I_i] =_{i=1}^{m} I_i = \mathcal{G}.$$

From this theorem it follows that, if I is a solvable ideal in $\mathcal{G}$, then $I = rad(g)$ if and only if $\mathcal{G}/I$ is semi-simple. Finally let us state without proof the theorem of Levi–Malcev. Let $\mathcal{G}$ be a Lie algebra. A Levi subalgebra of $\mathcal{G}$ is a Lie subalgebra which is a complement to $rad(\mathcal{G})$. It is a semi-simple algebra since it is isomorphic to $\mathcal{G}/rad(\mathcal{G})$. The theorem of Levi–Malcev says that, in every Lie algebra $\mathcal{G}$, there is a Levi subalgebra s. Therefore every Lie algebra decomposes as

$$\mathcal{G} = s + rad(g),$$

the sum of a semi-simple Lie algebra, and a solvable Lie algebra. This is the so-called Levi decomposition. ■

Exemple 2.2.8 *Let $\mathcal{G}$ be the Lie subalgebra of $M(n+1,\mathbb{R})$ consisting of the matrices*

$$\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} (x \in so(n), y \in \mathbb{R}^n)$$

(g is isomorphic to the Lie algebra of the motion group of $\mathbb{R}^n$). Let I be the ideal of g consisting of the matrices

$$\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}, (y \in \mathbb{R}^n)$$

It is Abelian, hence solvable. Let s be the subalgebra of g consisting of the matrices

$$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, x \in so(n)$$

If

$$n \geq 3,$$

the subalgebra s, which is isomorphic to $so(n)$, is semi-simple (even simple for $n = 4$). Therefore, since $\mathcal{G}/I \simeq s$, $I$is the radical of $\mathcal{G}$, and $\mathcal{G} = s + I$ is a Levi decomposition of $\mathcal{G}$.

Chapitre 3

Lie groups

3.1 Definitions and some proprieties.

The topological space locally looks like Euclidean space via suitable choice of coordinates,

$u_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is called a coordinate function.

$(a_1, ...a_n) \mapsto a_i$

Let M be a topological space and $p \in M$

Définition 3.1.1 *An m-dimensional chart at $p \in M$ is a pair (U, φ) where U is an open neighborhood of p and φ a homomorphism of U onto an open set in $\mathbb{R}^n$.*

The coordinates of the chart (U, φ) is $\varphi_i = u_i o \varphi$

$\mathbb{R}^n \overset{u_i}{\rightarrow} \mathbb{R}$

$\underset{U}{\uparrow} \quad \nearrow \varphi_i$ U is called a coordinate neighborhood and the pair (U, φ) a coordinate system, n is dimension of the chart (U, φ) .

Définition 3.1.2 *An m-dimensional topological manifold is a Hausdorff space with a countable basis such that $\forall p \in M$,there exist an m-dimensional chart at p.*

Remarque 3.1.3 *We can find a covering of M by open sets and each open set U in the covering is homeomorphic to the open m-ball $B_n = \{a \in \mathbb{R}^n, \|a\| \leq 1\}$*

A set $\mathcal{A}$ of charts of an m-dimensional manifold M is called a C$^\infty-$atlas if $\mathcal{A}$ satistifies the following conditions.

a) $\forall p \in M, \exists\ (U,\varphi) \in A$ and p$\in U$ i.e $M = \underset{(U,\varphi)\in A}{\cup} U$

b) (U,φ) and $(V,\psi) \in A$ then $U \cap V = \varnothing$ or the map $\varphi o \psi^{-1}$ and $\psi o \varphi^{-1}$ are of class C$^\infty$

We say that (U,φ) and (V,ψ) are compatible.

Définition 3.1.4 *Let $\mathcal{A}$ be a $C^\infty-$atlas on an m-dimensional manifold M, then the chart (U,φ) is admissible to $\mathcal{A}$ or compatible with $\mathcal{A}$ if (U,φ) is compatible with every chart in $\mathcal{A}$.*

Given any atlas $\mathcal{A}$, one can adjoin all chart which are admissible to $\mathcal{A}$ and obtain a collection $\bar{A}$ which is again an atlas on M.

$\bar{A}$ is maximal relative to properties a) and b).

An m-dimensional topological manifold M has C$^\infty-$differentiable structure or just a C$^\infty-$structure if one give M a maximal C$^\infty-$atlas.

A differentable manifold of class C^∞ or just $C^\infty-$manifold is an m-dimensional topological manifold M to which is assigned a C$^\infty-$differentiable structure.

Remarque 3.1.5 *One obtain differentable manifold of class C^∞ or real analytic manifold if the change of coordinate $\varphi o \psi^{-1}$ and $\psi o \varphi^{-1}$ is of class C^∞ or analytic.*

Basic Structures

Définition 3.1.6 *A Lie group is a set such that :*

(a) G is a group ;

(b) G is an analytic maniforld ;

(c) the group multiplication in (a) of the product manifold $\mu : G \times G \longrightarrow G : (x,y) \longmapsto xy$ and the group inversion operation in (a) ; $\tau : G \longrightarrow G : x \longmapsto x^{-1}$ are analytic functions relative to the structure in (b).

The dimension of a Lie group is the common dimension of all connected components of G.

Every Lie group is a topological group where the topology comes from its analytic structure because a manifold is a separate space and the map (x, y) $207bxy207b^1$ is continuous.

On the other hand, every Lie group is locally compact (a locally Euclidean manifold), locally connected, metrizable and complete.

A connected Lie group is countable at infinity and therefore separable.

1) The question of when a topological group is a Lie group is the fifth problem of Hilbert (posited in 1900).

The solution was given in 1956 by Montgomery and Zippin.

A locally Euclidean topological group is isomorphic to a Lie group. An interesting class of topological groups which are not Lie groups is the class of topological groups in infinite dimension which intervenes in quantum and classical physics. For example, the abelian group of Gauge transformations in electrodynamics, is not a Lie group because non locally Euclidean.

Let G be a locally compact group, and let F be the family of compact compact subgroups k of G such that the quotientG / k is a Lie group. Bruhat calls such a subgroup of G "a good subgroup".

The family $\mathcal{F}$ ordered by inclusion is decreasing filtering.

In fact, if k_1 and $k_2 \in \mathcal{F}$, the group $k_1/_{k_1 \cap k_2}$ $(\simeq G/_{k_1} \times k_1/_{k_1 \cap k_2})$ is an extension of the Lie group $G/_{k_1}$by the subgroup $k_1/_{k_1 \cap k_1}$ which is topologically isomorphic $k_1 k_2/_{k_2}$ because k_i sont compacts.

Since $k_1 k_2/_{k_2}$ is a Lie group, as a compact subgroup of the Lie group $G/_{k_2}$.

.

We deduce that $G/_{k_1 \cap k_2}$ is a Lie group, so $k_1 \cap k_2$ is a good subgroup.

Suppose $\cap \{k\} k \in \mathcal{F} = \{e\}$. This is the case, according to Montgomery and Zippin, if the quotient $G/_{G_0}$ of G by the connected connected component G_0 is compact. The group G is then canonically isomorphic to the projective limit of the Lie groups $G/_k$ for $k \in \mathcal{F}$. If, moreover, G is metrizable, there exists a decreasing sequence (k_n) of good subgroups such that $\cap k_n = \{e\}$. The group G is then canonically isomorphic to the projective limit of $G/_{k_n}$.

1) The additive groups R207f and the groups of matrices $GL\,(n, \mathbb{R})\,,\ S0\,(n, \mathbb{R})$, etc. are Lie groups.

2) If G is a discrete topological group, the neutral element e has an open neighborhood {e} homeomorphic to R2070 = {0}. A discrete topological group can be considered as a Lie group of dimension 0 and vice versa.

3) All closed subgroups of GL (n, R) are Lie groups. For example, the linear special group SL (n), othogonal groups (non-degenerate quadratic forms), symplectic groups (orthogonal group of non-degenerate alternating bilinear forms) and standard unipotent groups U_ {n}, ...

The map f induces an injective immersion $\widetilde{f} : G_1/_H \longrightarrow G_2$ which is homomorphism Lie group .

$$\begin{array}{ccc} G_1 & \overset{f}{\longrightarrow} & G_2 \\ p\downarrow & & \\ G_1/_H & \underset{\widetilde{f}}{\nearrow} & \end{array}$$

Let G be a connected Lie group. The underlying manifold admits a simply connected covering f : M 2192 G

Soit $\qquad \varphi : M \times M \longrightarrow G$

$$(m_1,\, m_2) \longrightarrow f\,(m_1)\ \ f\,(m_2)\ -\ \forall m_1,\, m_2 \in M.$$

Since $M \times M$ is simply connected, it is raised to a continuous mapping $\widetilde{\varphi} : M \times M \longrightarrow M$ such that $\widetilde{\varphi}\,(e,\, e) = e$ et $f \circ \widetilde{\varphi} = \varphi$ where e is chosen in M such that $f\,(e) = 1$.

$$\begin{array}{ccc} & M & \\ \overset{\widetilde{\varphi}}{\nearrow} & & \downarrow f \\ M \times M & \overset{\varphi}{\longrightarrow} & G \end{array}$$

where f is C^{∞}, F continuous and i an immersion .

For all $p \in P$, there exist an neigborhood U of p, an neigborhood V of $i(p) \in N$ and a map g of class C^{∞} such that $g \circ i = id/_U$. Since $f = i \circ F$, we have

$F = id \circ F = g \circ (i \circ F) = g \circ f$. Then F is C^{∞} as a composition of C^{∞} functions.

In particular, let put $F = f$ and $i = id$ we have the result.

Let G be a Lie group and H a submanifold of G which is also a subgroup of G. If H is a topological group (the topology is deduced from the analytic structure). Then H is a Lie subgroup

Since f is a covering, the kernel of f is necessarily a discrete invariant subgroup of M. Then there exists on M a Lie group structure for which f is a Lie group morphism. Thus, any connected Lie group G has a simply connected cluster. In other words, there exists a simply connected Lie group M and a surjective (discrete kernel) morphism f : M → G and therefore G is identified with the Lie group quotient M / D.

It is called the universal covering of G and is denoted by G.

1) The simply connected abelian Lie groups are the vector groups.

2) The Heisenberg groups, the unipotent groups,

The group ax + b are simply connected Lie groups ; In each case, the underlying topological space is an R ^{m}, where m is the dimension of the group.

Let G be a Lie group. A subset $H2282G$ is an analytic subgroup of G if

1) H is a subgroup of G

2) H is an analytic submanifold of G

Any analytic subgroup H of a Lie group G is a Lie group.

In fact :

The map f induces uniquely an injective immersion $\widetilde{f} : G_1/_H \longrightarrow G_2$ which is a morphisme of Lie group.

$$\begin{array}{ccc} G_1 & \xrightarrow{f} & G_2 \\ p \downarrow & & \\ G_1/_H & \underset{\widetilde{f}}{\nearrow} & \end{array}$$

Where f is C^∞, F is continuous and i is an immersion (P is a submanifold).

For any p2208P, there exists a neighborhood U of p, a neighborhood V of i (p) 2208 N and an application g of class C^∞ such that g 2218i = id / {U}.

Since H is a topological group, the map $f_H : H \times H \longrightarrow H$ is continuous. f_H is analytic then H is a Lie group.

Let G *be a Lie group and* G_0 *the identity component of* G. *Then*

a) G_0 *is an open invariant Lie subgroup of* G

b) - $T_eG = T_eG_0$ *and* $\dim G = \dim G_0$

c) $G/_{G_0}$ *is a discret Lie group.*

⋆ If G is connected Lie group and H a proper Lie subgroup then $\dim H < \dim G$.

⋆ We will show later that if G is a Lie group and if H is closed subgroup of G; then H is a Lie subgroup of G.

Hence $O(n)$, $SL(n)$ and $Sp(n)$ are closed Lie subgroup of $GL(n, \mathbb{R})$.

3.2 Lie Subgroups

We shall difine the concept of a Lie subgroup of a Lie group G and note that this concept differs from a topological subgroup because a Lie subgroup need not have the induced topolo

Définition 3.2.1 *Let* G *be a Lie group and let* H *be a Lie group. Then* H *is a Lie subgroup of* G *if* H *is an analytic submanifod of* G *and if* H *is a subgroup of* G.

Exemple 3.2.2 *(1) The torus T^2 is a Lie group when regarded as a product group $T^1 \times T^1$ and has as a submanifold the "irrational wrap around" curve as discussed in section 2.3 (see Fig. 4.1). This curve is given by $f(t) = (\exp 2\pi iat, \exp 2\pi ibt)$ where $a/b = \alpha$ is irrational. As we saw, this curve is a one-dimensional submanifold which is dense in T^2 and since $f(s+t) = f(s)f(t)$ in T^2, it is therefore a Lie subgoup which is not closed. This Lie subgroup does not have the induced topology since there points on the curve which are arbitrarily close in the topology induced from T^2 but are arbitrarily far apart in the topology of the curve.*

(2) The integrs $\mathbb{Z}$ are a zero-dimensional Lie subgroup of $\mathbb{R}$.

We shall now give various criterion for a subgroup to be a Lie subgroup and we need the following result [Helgason, 1962, P. 78] .

Lemme 3.2.3 *Let M an N be C^∞ (analytic) manifolds and let $f : M \longrightarrow N$ a C^∞ (analytic) mapping such that $f(M)$ is contained in a submanifold P. If the map $f : M \longrightarrow P$ is continous, then this map is also C^∞ (analytic).*

PROOF We shall schow this as follows. Let the accompanying diagram be commutative when f is C^∞, F continous and i an immersion (since P is a submanifold). Then by the remark following Proposition 2.23, for each $p \in P$, there is neighborhood U of p and a neighborhood V of $i(p) \in \mathbb{N}$, and a C^∞-map $g : V \longrightarrow U$ so that $g \circ i = identity \mid U$. Thus, since $f = i \circ F$, we have locally that

$$F = identity \circ F = g \circ (i \circ F) = g \circ f$$

and since the rigth side is a composition of C^∞-functions, F is C^∞. In particular, let $F = f$ and i be the identity ; that is, letting P be a submanifold, we obtain the result.

Proposition 3.2.4 *Let G be a Lie group and let H be a submanifold of G which is also an abstract subgroup of G. If H is also a topological (relative to the topology induced from its analytic structure), then H is a Lie subgroup of G.*

PROOF It suffices to schow H is a Lie group and this follows from the preceding lemma. The mapping

$$\begin{array}{rccc} f: & G \times G & \longrightarrow & G \\ & (x, y) & \longmapsto & xy^{-1} \end{array}$$

is analytic and its restriction $f_H : H \times H \longrightarrow G$ is continous. Thus by lemma 4.9, f_H is analytic so that H is a Lie group.

The next result follows from previous facts.

Proposition 3.2.5 *Let G be a Lie group and let H be a connected topological subgroup of G. Then there is at most one analytic structure $A(H)$ on H which makes H into a Lie subgroup of G.*

We now give some computional results which determine Lie subgroups.

Proposition 3.2.6 *Let H be a Lie group which is an abstract subgroup of the Lie group G. Assume at the identity $e \in G$ there exist an analytic chart (U, x) in G and an analytic chart (V, y) at e in H such that $x_i \mid H = y_i$ and $(\dfrac{\partial y_i}{\partial y_j}(e))$ has rank equal to the dimension of H. Then H is a Lie subgroup of G.*

PROOF We first translate the Charts at e to any point a by the analytic diffeomorphism $L(a)$ (of H and G) so that we can now apply corollary 2.12 to obtain$H$ is a submanifold.

This result can also stated in term of local Lie groups which generate a subgroup.

Corollaire 3.2.7 *Let G be a Lie group and let B be a local Lie group relative to the group operations in G. If there exist charts (U, x) at e in G and (V, y) at e in B such that $x_i \mid B = y_i$ and rank $(\dfrac{\partial y_i}{\partial y_j}(e)) = \dim B$, then the subgroup H generated by B is a connected Lie subgroup of G.*

We now note that the topological and manifold structure is mostly in the identity component. This will also become more evident when the Lie algebras are also taken into consideration.

Proposition 3.2.8 *Let G be a Lie group and G_0 be the identity component of G (as a topological group). Then :*

(a) G_0 is an open normal Lie subgroup of G ;

(b) $T(G, e) = T(G_0, e)$, and therefore $\dim G = \dim G_0$;

(c) G/G_0 is a Lie group which is discrete.

PROOF Since G is localy Euclidian, it has a connected open neighborhood U of e in G and from Proposition 3.23, U generates a connected subgroup H of G. Since G_0 is the identity component, we have by maximality that $H \subset G$. However H contains the neighborhood e of G_0 and G_0 is connected. Therefore $G_0 = H$. Now G_0 contains the neighborhood U of e and U is open in G so that any $a \in G_0$ is in the open neighborhood aU. Thus G_0 is open in G. This means G_0 is an open submanifold of G so that $\dim G = \dim G_0$ and $T(G, e) = T(G_0, e)$. Also by Theorem 3.22, G/G_0 is discrete.

Remarque 3.2.9 *(1) If G is a connected Lie group and H a proper Lie subgroup, them $\dim H < \dim G$ for otherewise H contains an open nucleus of G which generates G ; that is, $G = H$.*

(2) We shall show later that if G is a Lie group and if H is a closed subgroup of G, then H is a Lie subgroup of G. Thus the previously discussed subgoups $O(n)$, $SL(n)$ and $Sp(n)$ are all closed Lie subgroups of $GL(n, \mathbb{R})$.

(3) We shall consider later normal Lie subgroups when we discuss homomorphisms.

3.3 The Lie Algebra of a Lie group

We have seen from previous examples that the tangent space $T(G, e)$ of a Lie group G can be used to give local informations about G. In this chapter, we formalize this situation by introducing the G-invariant vector fields (G) and seeing that it is a vector space which is isomorphic to $T(G, e)$. Also (G) is a Lie algebra over $\mathbb{R}$ and induces a Lie algebra structure on $T(G, e)$.

Using this we define the exponential map exp : $(G) \longrightarrow G$ in terms of homomorphisms of $\mathbb{R}$ into G. The exponential map is a local diffeomorphism $\exp : U_0 \longrightarrow U_e$ of a suitable neighborhood U_0 of 0 in (G) onto a neighborhood U_e of e in G. Using the inverse function $\log : U_e \longrightarrow U_0$ we define canonical coordinates (U_e, log) at e in G. Thus by the action $L(a) : G \longrightarrow G : x \longmapsto ax$, we obtain coordinates at any point $a \in G$.

The exponential map is used to obtain a local representation of the multiplication in G analogous to the results of the Section 1.6. Thus for X and Y sufficiently near 0 in (G) we can write $\exp X \exp Y = \exp F(X,Y)$ where $F : (G) \times (G) \longrightarrow (G)$ is analytic at $(0,0) \in (G) \times (G)$. We show that the terms $F^{(k)}(0,0)(X,Y)^{(k)}$ of the Taylors series for F are in the subalgebra of (G) generated by X and Y. We briefly discuss the actual formula for $F(X,Y)$ which is known as the Campbel-Hausdorff formula. Finaly we show that a continous homomorphism of a Lie group is analytic. This yields the fact that the analytic structure of a Lie group is uniquely determined by its topology.

We now introduce the Lie algebra of a Lie group G in terms of invariant vector fields. Thus the Lie algebra will be determined by the tangent space $T(G,e)$ and the action of G detemines the values of the vector fields at any other point in G.

Définition 3.3.1 *An analytic vector field $X \in D(G)$ defined on a Lie group G is called invariant if for all $a \in G$ we have $[(TL(a))(e)]\, X(e) = X(a)$.*

Thus as in Section 2.7 we have since $(TL(a)(e)) : T(G,e) \longrightarrow T(G,a)$, then the value $[(TL(a))(e)]\, X(e)$ actually equals $X(a)$.

Next we note that if X is invariant, then X is $L(a)$-invariant for all $a \in G$; that is X is actually G-**invariant** or **left invariant** according to Section 2.7. For let $p \in G$, then

$$\begin{aligned} X(L(a)p) &= X(ap) = [(TL(ap)(e)]\, X(e) \\ &= [T(L(a) \circ L(p)(e)]\, X(e) \\ &= [TL(a)(p)] \cdot (TL(p)(e))(X(e)) \\ &= TL(a)(p) \cdot X(p) \end{aligned}$$

which gives the results.

Proposition 3.3.2 *Let G be a Lie group, let $X \in T(G,e)$ and let*

$$\widetilde{X} : G \longrightarrow T(G) : p \longmapsto \widetilde{X}(p),$$

where $T(G)$ is the tangent bundle of G with projection map π and $\widetilde{X}(p)$ is given by

$$(\widetilde{Xf})(p) = X(f \circ L(p))$$

where f is any real-valued analytic function on G. Then $\widetilde{X}$ is a G-invariant analytic vector field on G such that $\widetilde{X}(e) = X$. Furtheremore $\widetilde{X}$ is the unique G-invariant vector field on G such that $\widetilde{X}(e) = X$. Thus any G-invariant vector field on G is of the form $\widetilde{X}$.

PROOF Letting $TL(p) = TL(p)(e)$ we first note that $(\widetilde{X}f)(p) = (TL(p)X)(f)$ so that $\widetilde{X}(p) \in T(G,p)$ and therefore $(\pi \circ \widetilde{X})(p) = p$. Thus $\widetilde{X}$ is a vector field on G. Since $\widetilde{X}(p) = TL(p)X$, we have $\widetilde{X}(e) = X$ and the above computations show $\widetilde{X}$ is G-invariant. For the uniqueness, we use Proposition 2.34 with $f(a) = L(a)$ for any $a \in G$ or directly as follows. Let Z be a G-invariant vector field with $Z(e) = X$. Then $Z(p) = TL(p)(e)Z(e) = TL(p)(e)X = \widetilde{X}(p)$. Finally we shall show $\widetilde{X}$ is analytic and derive an other formula for it. Thus let $\alpha : I \longrightarrow G : t \longmapsto \alpha(t)$ be an analytic curve on an interval I containing $0 \in \mathbb{R}$ so that $\dot{\alpha}(0) = X \left[= \widetilde{X}(e)\right]$ and $\alpha(0) = e$. Then analogous to the results in section 2.7 we use the sesults on curves in section 2.5 to obtain

$$\begin{aligned}(\widetilde{X}f)(p) &= X(f \circ L(p)) \\ &= \frac{d}{dt}(0)(f \circ L(p) \circ \alpha) = \frac{d}{dt}\left[f(p\alpha(t)\right]_{t=0} \end{aligned} \qquad (*)$$

where $p\alpha(t)$ is the analytic product on G. Thus since f,α and the multiplication on G are analytic we have $\widetilde{X}f$ is an analytic function ; that is, $\widetilde{X}$ is analytic.

Let (G) denote the set of G-invariant vector fields on G. Then from the above result we see that (G) consists of all vector fields of the form $\widetilde{X}$ for $X \in T(G,e)$.

From $\widetilde{X}(p) = TL(p)(e)$ and $TL(p)(e)$ being injective we obtain the following.

Corollaire 3.3.3 *The map $\phi : (G) \longrightarrow T(G,e) : \widetilde{X} \longmapsto X$ is a vector space isomorphism. In particular the dimension of (G) over $\mathbb{R}$ equals the dimension of G and is finite.*

Corollaire 3.3.4 *(G) is a Lie algebra relative the bracket operation*

$$\left[\widetilde{X},\widetilde{Y}\right]=\widetilde{X}\widetilde{Y}-\widetilde{Y}\widetilde{X}.$$

Soient $\widetilde{X}$ et $\widetilde{Y}$ deux champs de vecteurs invariants.
Si $\widetilde{X}$ et $\widetilde{Y} \in L(G)$ alors $\left[\widetilde{X},\widetilde{Y}\right] \in L(G)$.

Preuve :
Il suffit de montrer que $\left[\widetilde{X},\widetilde{Y}\right]$ est invariant à gauche.
Soients a$\in G$ et $\varphi \in A(a)$. On a

$$\begin{aligned}
[dL(a)]\left[\widetilde{X},\widetilde{Y}\right]_e\varphi &= \left[\widetilde{X},\widetilde{Y}\right]_e(\varphi\circ L_a)\\
&= \widetilde{X}_e\left(\widetilde{Y}(\varphi\circ L_a)\right) - \widetilde{Y}_e\left(\widetilde{X}(\varphi\circ L_a)\right)\\
&= \widetilde{X}_e\left(\left(\widetilde{Y}\varphi\right)\circ L_a\right) - \widetilde{Y}_e\left(\left(\widetilde{X}\varphi\right)\circ L_a\right)\\
&= d\,L(a)\,\widetilde{X}_e\left(\widetilde{Y}\varphi\right) - d\,L(a)\,\widetilde{Y}_e\,(X\varphi)\\
&= X_a\left(\widetilde{Y}\varphi\right) - \widetilde{Y}_a\left(\widetilde{X}\varphi\right) = \left[\widetilde{X},\widetilde{Y}\right]_a\varphi.
\end{aligned}$$

d'où $d\,L_a\left[\widetilde{X},\widetilde{Y}\right]_e = [X,Y]_a$.

L'ensemble $\mathcal{L}(G)$ des champs de vecteurs invariants à gauche est donc une sous-algèbre de Lie de l'agèbre des champs de vecteurs C^∞. $L(G)$ **est appelé algèbre de Lie de** G. On la note $\mathcal{G}$.

T_eG est aussi muni d'une structure d'algèbre de Lie en posant pour tous

$$X, Y \in T_e(G), \quad [X,Y] = \left[\widetilde{X},\widetilde{Y}\right]_e$$

Par conséquent l'application $\phi : \mathcal{L}(G) \longrightarrow T_eG$

$$\widetilde{X} \longrightarrow X$$

est un isomorphisme d'algèbre de Lie.

L'algèbre de Lie T_eG est aussi appelé l'algèbre de Lie de G.

Exemple 3.3.5 *:*

Dans les paragraphes suivants, nous allons voir comment déterminer l'algèbre de Lie des groupes de Lie linéaires.

Soit $G = GL(n, \mathbb{R})$.

L'application $\phi : \mathcal{L}(G) \longrightarrow T_I(G)$ est un isomorphisme d'algèbre de Lie où le produit est $[X, Y] = \left[\widetilde{X}, \widetilde{Y}\right]_g (I)$ dans $T_I(G)$. On montre que $\mathcal{L}(G)$ est isomorphe à $gl(n, \mathbb{R})$.

Définition 3.3.6 *(a) The Lie algebra of a Lie group G is the Lie algebra (G) of G-invariant vector fields of G.*

(b) The Lie algebra G with product $[\ \ ]_{\mathcal{G}}$ is homomorphic to the Lie algebra h with product $[\ \ ]_h$ if there is a vector space homomorphism $\phi : G \longrightarrow h$ such that $\phi[X, Y]_{\mathcal{G}} = [\phi X, \phi Y]_h$ for all X, $Y \in G$. If ϕ is a vector space isomorphism, then G and h are isomorphic Lie algebras.

By means of corollary 5.4 we can make $T(G, e)$ into a Lie algebra as follows. Let X,$Y \in T(G, e)$ and $\widetilde{X}$, $\widetilde{Y}$ as above. Then define the product

$[XY] = \left[\widetilde{X}, \widetilde{Y}\right](e)$ which is in $T(G, e)$ and makes t(G,e) into a Lie algebra. This yields the following.

Corollaire 3.3.7 *The map $\phi : (G) \longrightarrow T(G, e) : \widetilde{X} \longmapsto X$ is a Lie algebra isomorphism.*

Frequently the lie algebra $T(G, e)$ is also called the "Lie algebra of G".

Exemple 3.3.8 *(1) Let $G = GL(V)$ Then from corollary 5.6 we have the map $\phi : (G) \longrightarrow T(G, I)$ is a Lie algebra isomorphism using the product $[XY] = \left[\widetilde{X}, \widetilde{Y}\right](I)$ in $T(G, I)$. However, we have the Lie algebra $gl(V)$ attached to G and we now show that (G) is isomorphic to $gl(V)$ as Lie algebras. Recall from example (3), Section 2.5 that for each $A \in gl(V)$ we defined an element $\overline{A} \in T(G, I)$ by*

$$(\overline{A}h) = [Dh(I)]\,A,$$

h analytic at I. The map $gl(V) \longrightarrow T(G,I) : A \longmapsto \overline{A}$ is a vector space isomorphism. Thus we obtain a vector field $\widetilde{\overline{A}}$ in (G) and consequently a vector space isomorphism $gl(V) \longrightarrow T(G,I) : A \longmapsto \widetilde{\overline{A}}$. We now show this is a Lie algebra isomorphism ; that is $\left[\widetilde{\overline{A}}, \widetilde{\overline{B}}\right] = \widetilde{\overline{[A,B]}}$. Usually $T(G,I)$ and $gl(V)$ are considered the same and the overbar is omited as done before, but we shall not do this now. Let $p \in G$ and A, $B \in gl(V)$. Then using $L(p)A = pA$, the product in $End(V)$, we have for f analytic in G

$$
\begin{aligned}
(\widetilde{\overline{A}}\ \widetilde{\overline{B}}) &= \widetilde{\overline{A}}(\widetilde{\overline{B}}(f))(p) \\
&= \overline{A}(\widetilde{\overline{B}}(f)\circ L(p), && \texttt{definition of } \overline{X} \\
&= \left[D(\widetilde{\overline{B}}(f)\circ L(p))(I)\right]A, && \texttt{definition of } \overline{A} \\
&= \left[D(\widetilde{\overline{B}}(f)(p)\circ D(L(p))(I)\right]A, && \texttt{chain rule} \\
&= \left[D(\widetilde{\overline{B}}(f)(p)\right](pA), && L(p)\ \texttt{linear} \\
&= \lim_{t\longrightarrow 0}\tfrac{1}{t}\left[(\widetilde{\overline{B}}f)(p+tpA)-(\widetilde{\overline{B}}f)(p)\right] \\
&= \lim_{t\longrightarrow 0}\tfrac{1}{t}\left[\overline{B}(f\circ L(p+tpA)-\overline{B}(f\circ L(p)\right] \\
&= \lim_{t\longrightarrow 0}\tfrac{1}{t}\left\{\left[D(f\circ L(p+tpA))(I)\right]B-\left[D(f\circ L(p))(I)\right]B\right\} \\
&= \lim_{t\longrightarrow 0}\tfrac{1}{t}\left\{\left[Df(p+tpA)\right](pB+tpAB)-\left[Df(p)\right]pB\right\}, && \texttt{chain rule} \\
&= \lim_{t\longrightarrow 0}\tfrac{1}{t}\left[Df(p+tpA)(pB)-Df(p)(pB)\right]+\lim_{t\longrightarrow 0}\tfrac{1}{t}\left[Df(p+tpA)\right](tpAB) \\
&= D^2f(p)(pB,pA)+Df(p)(pAB)
\end{aligned}
$$

Interchangin A and B, subtracting the equations and using the fact that $D^2f(p)$ is symmetric we obtain

$$
\begin{aligned}
(\widetilde{A}\,\widetilde{B})(f)(p) &= Df(p)(pAB - pA) \\
&= Df(p)([A,B]) \\
&= [D(f \circ L(p))(I)]\,[A,B] \\
&= \overline{[A,B]}(f \circ L(p)) \\
&= \widetilde{\overline{[A,B]}}(f)(p)
\end{aligned}
$$

which proves the result.

Chapitre 4

Solvable Lie Groups and Algebras

We now start the structural development of Lie groups and algebras. First we define a Lie group to be solvable if it is solvable as an abstract goup. Then the "derivative" of these results we discuss solvable Lie algebras. Thus we show that a connected Lie group is solvable if and only if its Lie algebra is solvable.Finillly we discuss Lie's theorem which involves finding a common characteristic vector for a solvable Lie algebra of endomorphisms acting on a complex vector space. This eventually yields that the matrices representing a solvable Lie algebra of endomorphisms acting on a complex vector space can be put into triangular form by using a suitable basis of the vector space. Once again, all fields in this chapter will be assumed to be of characteristic zero.

4.1 Solvable Lie group

Let G be an abstract group and let A and B be subgroups of G. Then we have the following notation.

(1) We denote by (A, B) the subgroup of G generated by all elements $xyx^{-1}y^{-1}$ for $x \in A$, $y \in B$.

(2) If A is a normal subgroup of G, we writr $A \lhd G$ or $G \rhd A$.

Note that if A and B are nomal subgroups of G, then (A, B) is a normal subgroup of G.

Définition 4.1.1 *Let $G^{(1)} = (G, G)$ and define by induction $G^{(k+1)} = (G^{(k)}, G^{(k)})$. Then we have the sequence of normal subgroups*

$$G \rhd G^1 \rhd G^2 \rhd \ldots .$$

Thus G is solvable if this sequence is and terminates at $\{e\}$; that is there exists n so that $G^{(n)} = \{e\}$ and G is called solvable of length n.

From results of Lang [1965] we have the following theorem :

Théorème 4.1.2 *Let G be an abstract goup. Then the following are equivalent.*

(a) The group G is solvable.

(b) There is a finite sequence of subgroups $G = G_0 \rhd G_1 \rhd G_2 \ldots \rhd G_n = \{e\}$ such that G_k/G_{k+1} is commutative for $k = 0, 1, \ldots, n-1$.

PROOF First we observe that by induction each $G^{(k)}$ is a normal subgroup of G. Now assume (a). Then note from the definition of (G, G) that $G/G^{(1)}$ is commutative and by induction and definition, $G^{(i)}/G^{(i+1)}$ is also commutative. Thus we have (b) by taking $G_k = G^{(k)}$. Conversly, assume we have a descending sequence $G = G_0 \rhd G_1 \rhd G_2 \ldots \rhd G_n = \{e\}$ with G_i/G_{i+1} commutative. Then G/G_1 being commutative implies $xyx^{-1}y^{-1}G_1 = eG_1$ which yields $G_1 \supset G^{(1)}$. Now assume $G_k \supset G^{(k)}$; Then since G_k/G_{k+1} is commutative we see

$$G_{k+1} \supset (G_k)^{(1)} \supset (G^{(k)}, G^{(k)}) = G^{(k+1)}.$$

However, since $G_n = \{e\}$ we see $G^{(n)} = \{e\}$ which gives (a).

Corollaire 4.1.3 *(a) A subgroup H of a solvable group is solvable.*

(b) If G is a solvable group of length n and H a normal subgroupn, then G/H is a sovable group of length less than or equal to n.

(c) If G is a group and H is a normal solvable subgroup of length n such that G/H is solvable of length m, then G is solvable less than or equal to $n+m$.

PROOF (a) We just note that by induction $H^{(k)} \subset G^{(k)}$.

(b) Let $\overline{G} = G/H$.Then by induction we see that $(\overline{G})^{(k)} = \overline{G^{(k)}}$ using $\pi : G \longrightarrow G/H : x \longmapsto \overline{x} = xH$ is a homomorphism. Thus the series for G yields the series $\overline{G} \rhd \overline{G}^1 \rhd \overline{G}^2 \rhd ... \rhd \overline{G}^{(n)} = \{eH\}$.

(c) Note that from the series $\overline{G} \rhd \overline{G}^1 \rhd \overline{G}^2 \rhd ... \rhd \overline{G}^{(n)} = \{eH\}$ we obtain the series $G \rhd G^1 \rhd G^2 \rhd ... \rhd G^{(n)}$ and $G^{(n)} \subset H$. However, since H is solvable we have $H \rhd H^1 \rhd H^2 \rhd ... \rhd \{e\}$ and we put these two series together to see that G is solvable.

Définition 4.1.4 *Let G be a Lie group. Then G is a solvable Lie group if G is solvable as an abstract group.*

Théorème 4.1.5 *Let G be a Lie group. Then the folowing are equivalent.*

(a) The Lie group is solvable.

(b) There exists a finite sequence $G = G_0 \rhd G_1 \rhd G_2 ... \rhd G_n = \{e\}$ such that each G_k is a closed subgroup of G with G_k/G_{k+1} commutative for $k = 0, 1, ..., n-1$.

(c) There exists a finite sequence $G = G_0 \rhd G_1 \rhd G_2 ... \rhd G_r = \{e\}$ such that for $k = 0, 1, ..., r-1$, we have G_k/G_{k+1} is a connected one dimensional group or a discrete group.

PROOF To show (a) implies (b), we recall that if H is a normal subgroup of G, then its closure $\overline{H}$ is a closed normal subgroup of G. Next assume G is solvable so we obtain the series $G \rhd G^1 \rhd G^2 \rhd ... \rhd G^{(m)} = \{e\}$ and let $G_0 = G$ and $G_k = \overline{G^k}$. Then G_k are closed normal subgroups (and, therefore, Lie groups) such that $G_0 \rhd G_1 \rhd G_2 ... \rhd G_m = \{e\}$. Furthermore G_k/G_{k+1} is a Lie group whitch is commutative, for let $\pi : G \longrightarrow G/G_{k+1}$. Then since $\pi(G^{(k)})$ is commutative, we have $\overline{\pi(G^{(k)})}$ is commutative. However, since π

is continuous $\pi(G_k) = \pi(\overline{G^{(k)}}) \subset \overline{\pi(G^{(k)})}$, so that $G_k/G_{k+1} = \pi(G_k)$ is commutative.

The converse (b) implies (a) is clear. Also (c) implies (b) is clear, so it remains to show (b) implies (c). Thus le H_k be the connected component of the commutative Lie group G_k/G_{k+1}. Then by the result outlined in the exercice (1), section 6.5, we have for the G_k in (b) $G_k/G_{k+1} = H_k \times D_k$ where D_k is a discret group, and $H_k \cong R^{q(k)} \times T^{p(k)}$ by theorem 6.20.

Next let $\pi : G \longrightarrow G/G_{k+1}$. Then in the series for (b) we replace each of the terms G_k by the series [with $p = p(k), q = q(k)$]

$$\pi^{-1}(R^p \times T^q \times D_k) \rhd \pi^{-1}(R^{p-1} \times T^q \times D_k) \rhd ... \rhd \pi^{-1}(T^q \times D_k) \rhd \pi^{-1}(T^{q-1} \times D_k) \rhd ... \rhd \pi^{-1}(D$$

Then we obtain the series in (c)

4.2 Solvable Lie Algebras and Radicals

Let $\mathcal{G}$ be a finite dimensional Lie algebra over a field $\Bbbk$ and let h, k be subspaces of $\mathcal{G}$. Then we shall use the following notation.

(1) We denote by $[h,k]$ the subspace of $\mathcal{G}$ generated by all products $[x,y]$ for $x \in h$ and $y \in k$. In particular $\mathcal{G}^{(1)} = [\mathcal{G},\mathcal{G}]$ is a subalgebra of $\mathcal{G}$.

(2) If h is an ideal of $\mathcal{G}$, then we write $\mathcal{G} \rhd h$ or $h \lhd \mathcal{G}$. In paricular, note $\mathcal{G} \rhd \mathcal{G}^{(1)}$. A Lie algebra $\mathcal{G}$ is **abelian** or **commutative** if $\mathcal{G}^{(1)} = \{0\}$.

Définition 4.2.1 *Let $\mathcal{G}$ be a fnite-dimensional Lie algebra over $\Bbbk$, set $\mathcal{G}^{(1)} = [\mathcal{G}\mathcal{G}]$, and define by induction $\mathcal{G}^{(k+1)} = \left[\mathcal{G}'^{k)}, \mathcal{G}^{(k)}\right]$.*

From the Jacobi identity for $\mathcal{G}$ we obtain $\mathcal{G} \rhd \mathcal{G}^{(1)} \rhd \mathcal{G}^{(2)} \rhd ...$ and we call $\mathcal{G}$ **solvable** if there exists n with $\mathcal{G}^{(n)} = \{0\}$. The smallest such n is called the **length** of the solvable algebra $\mathcal{G}$.

Théorème 4.2.2 *Let $\mathcal{G}$ be a finite-dimensional Lie algebra over $\Bbbk$. Then the folowing are equivalent.*

(a) The algebra $\mathcal{G}$ is solvable.

(b) There exists a sequence $\mathcal{G} = \mathcal{G}_0 \rhd \mathcal{G}_1 \rhd \mathcal{G}_2 ... \rhd \mathcal{G}_r = \{0\}$ so that the quotient algebra $\mathcal{G}_k/\mathcal{G}_{k+1}$ is commutative. Each $\mathcal{G}_k$ can be taken to be an ideal in $\mathcal{G}$.

(c) There exists a finite sequence of subalgebras $\mathcal{G} = \mathcal{G}_0 \rhd \mathcal{G}_1 \rhd \mathcal{G}_2 ... \rhd \mathcal{G}_s = \{0\}$ such that $\dim \mathcal{G}_k/\mathcal{G}_{k+1} = 1$. In general $\mathcal{G}_k$ is not an ideal in $\mathcal{G}$ but only in $\mathcal{G}_{k-1}$.

PROOF The equivalence of (a) an (b) is similar to those for groups in section 10.1. Thus for example, if $\mathcal{G}$ is solvable, then take $\mathcal{G}_k = \mathcal{G}^{(k)}$ for $k = 1, ..., r$ to obtain the sequence in (b) and also note $\left[\mathcal{G}'^{k)}\mathcal{G}^{(k)}\right] = \mathcal{G}^{(k+1)}$ so the desireted quotient algebra is commutative.

Next assume (c) where we have $\mathcal{G}_k/\mathcal{G}_{k+1} = \Bbbk\overline{X} = \Bbbk X + \mathcal{G}_{k+1}$ since $\mathcal{G}_k/\mathcal{G}_{k+1}$ is one dimensional. Then, since $\left[\overline{X}\ \overline{X}\right] = 0$ we have $\mathcal{G}_k/\mathcal{G}_{k+1}$ is a commutative Lie algebra. Thus (c) implies (b). Conversly, if the sequence in (b) is such that $\mathcal{G}_k/\mathcal{G}_{k+1} = \Bbbk X_1 + ... + \Bbbk X_r + \mathcal{G}_{k+1}$ is commutative, then each suspace $h_i = \Bbbk X_1 + ... + \Bbbk X_i + \mathcal{G}_{k+1}$ is an ideal in $\mathcal{G}_k/\mathcal{G}_{k+1}$ for $i = 1, ..., r$. Thus the corresponding subspaces h_i generated by $\{X_1, ..., X_i\} \cup \mathcal{G}_{k+1}$ where $X_i + \mathcal{G}_{k+1} = \overline{X_i}$ is an ideal in $\mathcal{G}_k$. Thus we obtain a sequence $\mathcal{G}_k = h_r \rhd h_{r-1} \rhd ... \rhd h_1 \rhd \mathcal{G}_{k+1}$, so that the quoteint ideals are one dimensional and this yields (c).

The proof of the following is similar to corollary 10.3.

Corollaire 4.2.3 *Let $\mathcal{G}$ be a Lie algebra containig the Lie algebra* h.

(a) If $\mathcal{G}$ is solvable, then h *is solvable.*

(b) If $\mathcal{G}$ is solvable and h an ideal of $\mathcal{G}$, then $\mathcal{G}/h$ is solvable of length less than or equal to the length of $\mathcal{G}$.

(c) If h is a solvable ideal of $\mathcal{G}$ such that $\mathcal{G}/h$ is solvable, then $\mathcal{G}$ is solvable.

Théorème 4.2.4 *Let G be a Lie group with Lie algebra $\mathcal{G}$.*

(a) If G is solvable, then $\mathcal{G}$ is solvable.

(b) If G is connected and $\mathcal{G}$ is solvable, then G is solvable.

PROOF (a) If G is a solvable lie gruop, then we have a sequence $G = G_0 \rhd G_1 \rhd G_2 ... \rhd G_n = \{e\}$ with each G_K is a closed normal Lie subgroup so that G_K/G_{K+1} is commutative. Then we obtain the corresponding sequence $\mathcal{G} = \mathcal{G}_0 \rhd \mathcal{G}_1 \rhd \mathcal{G}_2 ... \rhd \mathcal{G}_r = \{0\}$ of ideals of $\mathcal{G}$ so that $\mathcal{G}_k/\mathcal{G}_{k+1}$ is a commutative Lie algebra.

(b) If G is connected and $\mathcal{G}$ is solvable, then we shall show G is solvable by induction on the length of $\mathcal{G}$. Thus let $\mathcal{G} \rhd \mathcal{G}^{(1)} \rhd ... \rhd \mathcal{G}^{(n-1)} \rhd \mathcal{G}^{(n)} = \{0\}$ be the sequence for $\mathcal{G}$ and let K be the Lie subgroup of G generated by $\exp(\mathcal{G}^{(n-1)})$. Then K is a commutative normal subgroup of G (since $\mathcal{G}^{(n-1)}$ is a commutative ideal of $\mathcal{G}$) and its closue $\overline{K} = H$ is also a commutative normal Lie subgroup of G. Now let h be the Lie algebra of H. Then h is a commutative lie algebra of $\mathcal{G}$ and $\mathcal{G}^{(n-1)} \subset h$. From this we have [exercicev (2) above], $\mathcal{G}/h \cong (\mathcal{G}/\mathcal{G}^{(n-1)})/(h/\mathcal{G}^{(n-1)})$ and since $\mathcal{G}/\mathcal{G}^{(n-1)}$ is solvable of length less than or equal to $n-1$ we have $\mathcal{G}/h$ is solvable of length less than or equal to $n-1$.(corollary 10.8). Thus by the induction hypotheses G/H is solvable and since H is solvable, we have G is solvable using corollary 10.3.

Lemme 4.2.5 *Let $\mathcal{G}$ be a finite-dimensional Lie algebra over $\Bbbk$. Then there exists a unique maximal sovable ideal of $\mathcal{G}$/ namely the sum of the solvable ideals of $\mathcal{G}$. This solvable maximal ideal is called the **radical** of $\mathcal{G}$ and is denoted by r. Moreever $\mathcal{G}/r$ is $\{\overline{0}\}$ or contains no proper solvable ideals ; that is the **radical** of $\mathcal{G}/r$ is $\{\overline{0}\}$.*

PROOF Let h and k be solvable ideals of $\mathcal{G}$. Then the vector subspace $h+k$ is an ideal of $\mathcal{G}$. Now by the above exercicise (2) we see $(h+k)/k \cong h/(h \cap k)$ and since $h \cap k \subset h$ is solvable we have $h/(h \cup k)$ is solvable ; Thgus we have $(h+k)/k$ is solvable and k is solvable so that by corollary 10.8 $h+k$ is solvable Thus since $\mathcal{G}$ is finite dimensional, the solvable ideal of maximim dimension is unique and by the above, contains every solvable ideal of $\mathcal{G}$; denote this maximal solvable ideal by r.

Next let $\overline{h} = h/r$ be solvable ideal of $\overline{\mathcal{G}} = \mathcal{G}/r$ where h is some ideal of $\mathcal{G}$ with $h \supset r$. Then since h/r is solvable and r is solvable we have by corollary 10.8 that h is solvable. Thus $h \subset r$, so that $\overline{h} = \{\overline{0}\}$.

Définition 4.2.6 *Let G be a Lie group with lie algebra $\mathcal{G}$ and let r be the radical of $\mathcal{G}$. Then we define the **radical** of G, $R = rad\ G$, to be the connected Lie subgroup of G whose Lie algebra is $r = rad\ \mathcal{G}$.*

Proposition 4.2.7 *Let G be a Lie group with radical R. Then R is closed and R is the maximal solvable normal connected Lie subgroup of G.*

PROOF Let $\overline{R}$ denote the closure of R. Then $\overline{R}$ is a normal, solvable Lie subgroup (since it is closed) ; Thus its Lie algebra $\overline{r}$ is solvable (Theorem 10.9) so that $r = \overline{r}$ and consequently $\overline{R} = R$; that is R is closed, normal, solvable Lie subgroup of G. The fact that R is maximal among connected Lie subgroups with those proprerties also uses the maximally of r.

Corollaire 4.2.8 *The radical of $G/R = \{eR\}$.*

Définition 4.2.9 *(a) A finite-dimensional Lie algebra is called **semi-simple** if it has no proper solvable ideal. Thus $\mathcal{G}$ is semi-simple if $r = \{0\}$. Similary a Lie group G is **semisimple** if its radical $R = \{e\}$.*

*(b) A Lie group G is **simple** if its Lie algebra $\mathcal{G}$ is **simple**. That is $[\mathcal{G}\mathcal{G}] \neq \{0\}$ $\mathcal{G}$ has no proper ideals.*

We shall eventually show that a semisimple Lie algebra over a field of characteristic 0 is a direct sum of simple lie algebras which are ideals. Consequently many problems involving semisimple Lie groups can be done in terms of simple Lie algebras.

4.3 Lie's Theorem on Solvability

We now describe how a solvable Lie group or Lie algebra of endomorphisms can be represented by triangular matricies. To do this we compute characteristic foots so we consider real Lie groups or algebras acting on complex vector spaces.

Définition 4.3.1 *(a) Let $\mathbb{k}$ be a field of characteristic 0 and let V be a finite-dimensional vector space over $\mathbb{k}$. Let $T \in End_{\mathbb{k}}(V)$ and $\lambda \in \mathbb{k}$. Then $V_\lambda = \{X \in V : TX = \lambda X\}$ and $V(\lambda) = \{X \in V : (T - \lambda I)^n X = 0$ for some $n \in N\}$ where N is the set of natural numbers (which are greater than 0). If $V_\lambda \neq \{0\}$ then λ is called a* ***characteristic value*** *or* ***eigenvalue*** *of T and $0 \neq X \in V_\lambda$ is called an* ***eigen vector*** *or* ***characteristic vector*** *of T with characterisctic value λ. If $V(\lambda) \neq \{0\}$, then λ is called a weight of T and $V(\lambda)$ a* ***weight space*** *and $0 \neq X \in V(\lambda)$ is called a* ***weight vector*** *of T*

A characteristic value or a weight λ of T is a solution of the equation $\det(Ix - T) = 0$ *and if the solutions of this (characteristic) equation are in* $\mathbb{k}$, *then we say that the* **characteristic values or weights are in $\mathbb{k}$** ; *recall the definition of a split endomorphism in section 9.2.*

(b) Let $N \subset End_{\mathbb{k}}(V)$, let $f : N \longrightarrow \mathbb{k}$ be a function, and set

$V_f = \{X \in V : \text{for all } T \in N, TX = f(T)X\}$ *and*

$V(f) = \{X \in V : \text{for all } T \in N, \text{ there exits } n > 0 \text{ with } (T - f(T)I)^n = 0\}$.

If $V_f \neq \{0\}$, then f is called characteristic function on N and $0 \neq X \in V_f$ is called a characteistic vector of N for the characteristic fucnction f. Similary one defines a ***weght function***, ***weight space*** *and* ***weight vector*** *in case $V(f) \neq \{0\}$.*

Thus these functions on N asign to each T in N a characteristic root $f(T)$ of T. Of course, in actual computations, the characteristic roots discussed above might be in the algebraic closure of $\mathbb{k}$.

with these definitions and results on canonical forms of endomorphisms [Jacobson,1953,Vol.II ;Lang,1965] we state the following :

Proposition 4.3.2 *Let V be a finite-dimensional vector space over K and let $T \in End_{\mathbb{k}}(V)$ have its (distinct) weights $\lambda_1, \lambda_2, ..., \lambda_m$ in $\mathbb{k}$. Then the weight spaces $V(\lambda_i)$ are T-invariant and $V = V(\lambda_1) + V(\lambda_2) + ... + V(\lambda_m)$ (direct sum).*

Remarque 4.3.3 *This direct sum decomposition will be generalized in the next chapter to a direct sum decompostion of weight spaces of a nilpotent*

Lie group or Lie algebra.

Proposition 4.3.4 *Let V be a finite-dimensional vector space over $\mathbb{R}$, let G be a Lie subgroup of $GL(V)$, and let $f : G \longrightarrow \mathbb{R}$ be a characteristic function with ; $f(G) \subset \mathbb{R}^* = \mathbb{R} - \{0\}$. Then regarding $\mathbb{R}^*$ as a multiplicative Lie group, the map $f : G \longrightarrow \mathbb{R}^*$ is an analytic homomorphism of Lie groups. f is frequently called a character of G.*

PROOF Let S,T $\in G$. Then for $0 \neq X \in V_f$ we have $SX = f(S)X$ and $T(X) = f(T)X$. Thus $STX = Sf(T)X = f(T)SX = f(T)f(S)X$.

However since $(ST)X = f(ST)X$ this gives $f(ST) = f(S)f(T)$ so that $f : G \longrightarrow \mathbb{R}^*$ is a homomorphism. To see that f is analytic, let $X_1, X_2, ..., X_m$ be a basis of V so X_1 is a charateristic vector of G for the characteristic function f. Noting the mappings $r : G \longrightarrow V : S \longmapsto S(X_1)$ and $s : V \longrightarrow \mathbb{R} : \sum_{i=1}^{n} \lambda_i X_i \longmapsto \lambda_1$ are analytic, so is the map $f = s \circ r : G \longrightarrow \mathbb{R}^*$.

Analogous to the lemma 7.15 we are the following result :

Lemme 4.3.5 *Let V be a finite-dimensional vector space over $\mathbb{R}$ and let G be a real connected solvable Lie group which is a subgroup of $GL(V)$ and has real Lie algebra. Let W be a subspace of V.*

(a) W is invariant under the action of G if and only if W is invariant under the action of $\mathcal{G}$.

(b) For $A \in G$, the vector $X \in V$ is a characteristic vector of A with characteristic value λ if and only if X is characteristic vector of the group $\{\exp tA : t \in \mathbb{R}\}$ for the characteristic function $f : \exp tA \longmapsto e^{t\lambda}$.

The following result or some of its equivalent conseqences is known as " Lie's theorem on sovability". We follow the work of Tits [1965] for the group proof.

Théorème 4.3.6 *(Lie's theorem). Le V be finie-dimensioal vector space over $\mathbb{C}$ and let G be a real connected solvable Lie group which is a subgroup of $GL(V, \mathbb{C})$. Then there exists a nonzero characteristic vector of G for some characteristic function.*

PROOF We shall proof the results by induction on the dimension of G.

First, if G is one dimensional and $0 \neq A \in \mathcal{G}$ which is the Lie algebra of G, then since $\mathcal{G} \subset gl(V)$ we see that A has a nonzero characteristic vector $X \in V$. However by lemma 10.18 and exercise (2), X is also a characteristic vector of G. Net assume G is of dimension n and assume as an induction hypothesis that we have shone the result for all such groups of smaller dimension. Now since G is connected and solvable, G has a connected solvable normal subgroup H of dimension $n-1$ [Exercice (5), section 10.2]. Thus by the induction hypothesis we can conclude there is a characteristic function $f : H \longrightarrow \mathbb{C}^* = \mathbb{C} - \{0\}$ and analogous as Proposition 10.17 we have f is continous.

We shall now show that the subspace $V_f = \{X \in V : SX = f(S)X \text{ for all } S \in H\}$ is invariant under G. Thus let $X \in V_f$, $S \in H$ and $T \in G$. Then

$$S(T(X)) = (ST)(X) = T(T^{-1}ST)(X) = f(T^{-1}ST)T(X) \qquad (\star)$$

using $T^{-1}ST \in H$. Thus the number $f(T^{-1}ST)$ is a characteristic value of S with characteristic vector $T(X)$ and also the fonction $k : G \longrightarrow \mathbb{C}^* : T \longmapsto f(T^{-1}ST)$ is continous; However, since G is connected and the set of characteristic values of S is discrete, the image $k(G)$ consists of a single point. (This uses the characterization : the topological space M is connected if and only if M is mapped continously into a discrete space implies the image of M consists on a single point.) Thus we have $k(T) = k(I) = f(S)$ Using this in $(\star)$ we have for any $X \in V_f$, $T \in G$, and $S \in H$ that $S(T(X)) = f(S)T(X)$ which shows by the definition of V_f that V_f is invariant under the action of G.

Next by lemma 10.8 we have that V_f is invariant under the action of $\mathcal{G} \subset gl(V)$. Therefore if $A \in \mathcal{G}$ and $A \notin h$ which is the Lie algebra of H, then, since the subspace V_f is invariant under the linear transformation A, there exists a characteristic vector $0 \neq V_f$ for A. Thus since $\mathcal{G}/h = \mathbb{R}A + h$ (using the hypothesis that $\dim H$ is $n-1$) we see that X is a characteristic vector for $\mathcal{G}$. For let $B = aA + bC \in \mathcal{G}$ with $C \in h$ and for $AX = \lambda X \in V_f$

and $CX = \mu X$ [Using lemma 10.8 (b) applied to h and H] we have $BX = (aA+bC)X = (a\lambda+b\mu)X$. Thus by lemma 10.18, X is a characteristic vector of G.

Définition 4.3.7 *Let V be an $m-$dimensional vector space other the field $\mathbb{k}$. Then a sequence of subspaces $\{0\} \subset V_1 \subset V_2 \subset ... \subset V_m = V$ such that $\dim V_i = i$ for $i = 1,..,m$ is called a **flag** of V. Let $G \subset GL(V)$ be a Lie group. Then the flag is G-invariant if for every $T \in G$ we have $T(V_i) \subset V_i$ for $i = 1,...,m$. Similarly for a Lie algebra G of endomorphisms, we define a G-invariant flag.*

Proposition 4.3.8 *Let V be an n-dimensional vector space over $\mathbb{C}$ and let G be a real connected Lie group which is a subgroup of $GL(V, \mathbb{C})$. Then the following are equivalent.*

(a) The group G is solvable.

(b) There exists a flag which is G-invariant.

(c) There exists a flag of V such that the matrices for the elements in G can be put simultaneously into triangular form. (The matrices might have compex entries).

PROOF Assume G is solvable. Then to show (b) we use induction on the dimension of V. From Lie's theorem there is a one-dimensional subspace W of V which is invariant under G. Therefore an element $T \in G$ induces a nonsingular linear map

$\overline{T} : V/W \longrightarrow V/W : x + W \longmapsto Tx + W$ and the map

$G \longrightarrow GL(V/W, \mathbb{C}) : T \longmapsto \overline{T}$ is an analytic homomorphism. Thus the image $\overline{G} = \left\{\overline{T} \in GL(V/W, \mathbb{C}) : T \in G\right\}$ is a real connected solvable Lie group which is a subgroup of $GL(V/W, \mathbb{C})$ and by the induction hypoyhesis there exists a flag in V/W which is invariant under $\overline{G}$ $\{\overline{0}\} \subset \overline{V}_2 \subset \overline{V}_3 \subset ... \subset \overline{V} = V/W$. Now let $\pi : V \longrightarrow V/W$, let $V_i = \pi^{-1}(\overline{V}_i)$, and set $V_1 = W = \pi^{-1}(\{\overline{0}\})$. Then $\dim V_i = i$ and $\{O\} \subset V_1 \subset V_2 \subset ... \subset V_n = V$ is a flag which is invariant under G.

Next to show (b) implies (c) we choose a basis of V from the corresponding flag as follows. Let $V_1 = \{X_1\}$. Then since $TV_1 \subset V_1$ for $T \in G$ we have $TX_1 = a_{11}(T)X_1$. Next let $V_2 = \{X_1, X_2\}$ where X_1 and X_2 are independent using $\dim(V_2/V_1) = 1$. Then since $TV_2 \subset V_2$ we have $TX_2 = a_{12}(T)X_1 + a_{22}(T)X_2$ for all $T \in G$. Continuing in this manner we can choose a basis of V so that any $T \in G$ has a matrix of the form

$$\begin{bmatrix} a_{11}(T) & a_{12}(T) & . & . & . & a_{1n}(T) \\ & a_{22}(T) & & & & \\ & & . & & & \\ & & & . & & \\ & & & & . & \\ 0 & & & & & a_{nn}(T) \end{bmatrix}$$

with $0 \neq a_{11}(T)...a_{nn}(T) = \det T$.

Finaly to show (c) implies (a) let G be represented by the group triangular matrices as above; Let G_1 be the normal subgroup of triangular matrces of the form

$$\begin{bmatrix} 1 & & & & * \\ 0 & 1 & & & \\ & & . & & \\ & & & . & \\ & & & & . \\ 0 & & ... & 0 & 1 \end{bmatrix}$$

with $1's$ on the diagonal. Let G_2 be the normal subgroup of G_1 of the form

$$\begin{bmatrix} 1 & 0 & & & * & \\ & 1 & 0 & & & \\ & & 1 & & & \\ & & & . & & \\ & & & & . & 0 \\ 0 & & & & & 1 \end{bmatrix}$$

with $1's$ on the diagonal and $0's$ on the superdiagonal. Continuing this way we obtain the sequence $G \rhd G_1 \rhd G_2 ... \rhd G_n = \{I\}$ with G_i/G_{i+1} commutative.

The preceding results on Lie groups can be translated into results on Lie algebras via the exp mapping or directly as follows. This proof involves some computations we shall see again in chapter 11.

Théorème 4.3.9 *Let P be the algebraic closure of the field $\mathbb{k}$ and let V be a nonzero finite-dimensional vector space of P. Let $\mathcal{G}$ be a solvable Lie algebra over $\mathbb{k}$ and let ρ be a homomorphism of $\mathcal{G}$ into $gl(V, P)$. Then there exists a vector $0 \neq X \in V$ which is a characteristic vector for all the members of $\rho(\mathcal{G})$ for some characteristic funtion.*

PROOF We prove this by induction on the dimensional of $\mathcal{G}$. For $\dim \mathcal{G} = 1$,the theorem follows from the results on canonical form (Proposition 10.16). We are assume the results hold for all Lie algebras of dimension less than $\dim \mathcal{G}$. From theorem 10.7 (d) we can find an ideal h in $\mathcal{G}$ so that $\dim \mathcal{G}/h = 1$. By corollary 10 we have h is solvable so that by the induction assumption there exists a characteristic function $f : h \longrightarrow P$ so that for all $S \in h$, $\varphi(S)X = f(S)X$. From $\dim \mathcal{G}/h = 1$ we can find $T \in \mathcal{G}$ so that $T \notin h$. Thus $\mathcal{G} = \mathbb{k}T + h$. Let W be the subspace of V spanned by all the vectors $X_1 = X$ and $X_{k+1} = \varphi(T)^k X$. for $k = 1, 2....$ Now note W is $\rho(T)$-invariant subspace of V.

We shall now show : For all $S \in h$, $\varphi(S)X = f(S)X$ for all $Y \in W$; that is, W is $\rho(T)$-invariant and furthermore $\rho(S) = f(S)I$ on W.

We first prove by induction that for all $S \in h$ and $k = 1, 2, ...$

$$\rho(S)X_k = f(S)X_k + a_{k-1}X_{k-1} + ... + a_1 X_I \qquad (*)$$

where $a_j = a_j(S)$ are in P. By the choice for $X_1 = X$ the result holds for $k = 1$. Assuming $(*)$ for k, we have

$$\begin{aligned}\rho(S)X_{k+1} &= \rho(S)\rho(T)X_k, \text{ definition of } X_{k+1}\\ &= \rho([ST])X_k + \rho(T)\rho(S)X_k\\ &= \rho([ST])X_k + \rho(T)(f(S)X_k + a_{k-1}X_{k-1} + ... + a_1X_I)\\ &= f(S)\rho(T)X_k + b_kX_k + ... + b_1X_1\\ &= f(S)X_{k+1} + b_kX_k + ... + b_1X_1\end{aligned}$$

using $ST \in h$ and the induction assumption.

We next prove $\varphi(S)X = f(S)X$ for all $Y \in W$. From $(*)$ and the definitiin of X_k we first observe that W is $\rho(T)$-invariant. Next note that from the above, the restriction $\rho(S) \mid W$ has matrix

$$\begin{bmatrix} f(S) & & & * \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & f(S) \end{bmatrix}$$

so that $tr(\rho(S) \mid W) = f(S)\dim W$ for $S \in h$. Next note that $\rho(S)$ and $\rho(T)$ map W into W so that $\rho([ST]) = \rho(S)\rho(T) - \rho(T)\rho(S)$ as endomorphisms of W. Since $tr(AB)) = tr(BA)$ for endomorphisms, we see $0 = \rho([ST]) = f([ST])\dim W$. Since $\dim W \geqslant 1$ this gives $f([ST]) = 0$. Thus

$$\begin{aligned}\rho(S)X_{k+1} &= \rho(S)\rho(T)X_k\\ &= \rho([ST])X_k + \rho(T)\rho(S)X_k\\ &= f([ST])X_k + f(S)\rho(T)X_k\\ &= f(S)X_{k+1}\end{aligned}$$

that is $\varphi(S)X = f(S)X$ for all $Y \in W$.

Since W is $\rho(T)$-invariant and P is algebraically closed we se that $\rho(T)$ has a characteristic vector $A \in W : \rho(T)A = tA$. Also $\rho(S)A = f(S)A$ for all $S \in h$ and since $\mathcal{G} = \Bbbk T + h$ we have for any $Z = aT + S$ that $\rho(Z)A = a\rho(T)A + \rho(S)A = (at + f(S))A$. Thus A is a characteristic vector of $\rho(\mathcal{G})$ and $F : aT + S \longmapsto at + f(S)$ defines the corresponding characteristic funtion.

The formalities in the proof of propositioin 10.21 yield the following :

Proposition 4.3.10 *Let P be the algebraic closure of the field $\Bbbk$ and let V a nonzero finite-dimensional vectoe space over P. Let $\mathcal{G}$ be a Lie algebra over $\Bbbk$ and let ρ be a homomorphism of $\mathcal{G}$ into $gl(V, P)$. Then the following are equivalent.*

(a) The Lie algebra $\rho(\mathcal{G})$ is solvable.

(b) There is a flag in $\mathcal{G}$ which is invariant under $\rho(\mathcal{G})$.

(c) There is a basis of V such that the matrices for the endomorphisms in $\rho(\mathcal{G})$ can be put simultaneously in the triangular form. (The matrices might have entries from P.)

These results apply when we take the fiel $\Bbbk$ to be algebraically closed itself. Thus $\Bbbk = P$ and we obtain the following :

Proposition 4.3.11 *Let $\mathcal{G}$ be a Lie algebra over the albebraically closed field $\Bbbk$. Then $\mathcal{G}$ is solvable if and only if thre is a flag in $\mathcal{G}$*

$\{0\} \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset ... \subset \mathcal{G}_n = \mathcal{G}$ *such that each $\mathcal{G}_i$ is an ideal of $\mathcal{G}$.*

PROOF Assume $\mathcal{G}$ is solvable. Then since $\mathcal{G} \longrightarrow ad(\mathcal{G}) : X \longmapsto adX$ is a homomorphism of Lie algebras over $\Bbbk$, we see that $ad(\mathcal{G})$ is a solvable Lie algebra of endomorphisms acting on the vector space $\mathcal{G}$. By proposition 10.23 (b) there is a flag $\{0\} \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset ... \subset \mathcal{G}_n = \mathcal{G}$ which is invariant under $ad(\mathcal{G})$; that is, each $\mathcal{G}_i$ is an ideal of $\mathcal{G}$.

Conversely, assuming such a flag exists we see that $ad(\mathcal{G})$ is solvable ; using (b) implies (a) in proposition 10.23. However, $ad : \mathcal{G} \longrightarrow ad(\mathcal{G})$ is a homomorphism so that $ad(\mathcal{G}) \approxeq \mathcal{G}/\ker(ad)$. Since $\ker(ad)$ is the center of $\mathcal{G}$ which is solvable and since $\mathcal{G}/\ker(ad)$ is solvable, we have by corollary 10.8 that $\mathcal{G}$ is solvable.

Chapitre 5

Nilpotent Lie groups and algebras

We continue the concepts given in the preceding chapter and call a Lie group nilpotent if it is nilpotent as an abstract group. Then we discuss nilpotent Lie algebras and obtain the result that a connected Lie group is nilpotent if and only if its Lie algebra is nilpotent. In the last section we consider the vector space decomposition which yiels the Jordan canonical form for an endomorphism and extend this compostion to a nilpotent group of automorphisms.

5.1 Nilpotent Lie Groups

We now give a variation of the results on solvable groups using some of the notation of the preceding chapter.

Définition 5.1.1 *Let G be an abstract group.*

(a) Let $C^0G = G$ and $C^{n+1}G = (G, C^nG)$. Then $C^nG \rhd C^{n+1}G$ and we have the ***descending central series*** $G = C^0G \rhd C^1G \rhd C^2G...$

(b) Let $C_0G = \{e\}$ *and let* $C_nG = \pi^{-1}(Z(G/C_{n-1}G)$ where $Z(G/C_{n-1}G$ *is the center of* $G/C_{n-1}G$ *noting by inductioin*

$C_nG \lhd C_{n+1}G \lhd G$ *and where* $\pi : G \longrightarrow G/C_{n-1}G$ *is the corresponding projection map. Thus we have the* **ascending central series**

$\{e\} = C_0G \lhd C_1G \lhd C_2G \lhd ....$

Théorème 5.1.2 *Let* G *be an abstrct group. Then the following are equivalent.*

(a) There exists a series of normal subgroups of G

$G = G_0 \rhd G_1 \rhd ... \rhd G_s = \{e\}$ *such that*

$(G, G_n) \subset G_{n+1}$ *for* $n = 0, ..., s-1$.

(b) There exists a positive integer p *such that*

$G \rhd C^1G \rhd ... \rhd C^pG = \{e\}$.

(c) There exists a psitive integer q *such that*

$\{e\} \lhd C_1G \lhd ... \lhd C_qG = G$.

PROOF Assume there is a series as given in (a). Then by induction we have $G_n \supset C^nG$. Thus $C^sG = \{e\}$. Conversely if (b) holds, then we automatically have a series satisfaying (a).

Next we have (a) implies (c), for if we have a series as in (a), then we shall show by induction $G_{s-n} \subset C_nG$ so that for $n = s$ we obtain $C_sG \subset G$. Thus $\{e\} = G_s \subset C_0G = \{e\}$ and assume $G_{s-1} \subset C_iG$. Then $(G/C_iG, G_{s-i-1}/C_iG) \subset G_{s-i}/C_iG \subset C_iG/C_iG = \{\overline{e}\}$ using the induction hypothesis for the second inclusion ; that is, $(G, G_{s-i-1}) \subset C_iG$. Thus if $\pi : G \longrightarrow G/C_iG$ is the projection, we se that

$G_{s-i-1} \subset \pi^{-1}(Z(G/C_iG)) = C_{i+1}G$, using the definition of $C_{i+1}G$.

Conversely, to see (c) impliies (a), we first note that

$(G, C_iG)/C_{i-1}G \subset (G/C_{i-1}G, C_iG/C_{i-1}G) = \{\overline{e}\}$ using $C_iG = \pi^{-1}(Z(G/C_{i-1}G))$, where $\pi : G \longrightarrow G/C_{i-1}G$. Thus $(G, C_iG) \subset C_{i-1}G$. So that for

$C_qG = G = G_0, C_{q-1}G = G_1, ..., C_{m-1}G = G_m$,etc, we see that the series in (c) yields the series in (a).

Définition 5.1.3 *An abstract group G is **nilpotent** if it satisfaties any one of the conditions of theorem 11.2.*

Remarks (1) Note that nilpotency involves a descending series commutators of the terms of the series with the group, whereas solvability involves descending series using commutators of the terms of the series with itself.

(2) Subgroups, quotient groups, and finite direct products of nilpotent groups are nilpotent. The proofs run as expected. For example, if G_i are groups with $C^{n_i}G = \{e_j\}$ for $i = 1, ..., m$ and if $G = G_1 \times ... \times G_m$, then $C^nG = \{e\}$, where $n = \max\{n_1, ..., n_m\}$.

Théorème 5.1.4 *Let G be a Lie group. The following are equivalent.*

(a) As an abstract group G is nilpotent.

(b) There is a series $G = G_0 \rhd G_1 \rhd ... \rhd G_s = \{e\}$ where each G_n is a closed normal Lie subgroup of G and $(G, G_n) \subset G_{n+1}$.

(c) If $\overline{C}^0G = G$ and $\overline{C}^{n+1}G = \overline{(G, \overline{C}^nG)}$, then there exists a positive integer p such that $G \rhd \overline{C}^1G \rhd ... \rhd \overline{C}^pG = \{e\}$.

PROOF Showing (c) if and if (b) is similar to theorem 11.2 ; (b) implies (a) is also clear. Next assume (a). Then there is a series of normal subgroups $G \rhd G_1 \rhd ... \rhd G_s = \{e\}$ with $(G, G_n) \subset G_{n+1}$. Consequently we obtain $\overline{G}_s = \{e\}$, using G as Hausdorff, and $G \rhd \overline{G}_1 \rhd ... \rhd \overline{G}_s = \{e\}$ with $(G, \overline{G}_n) \subset \overline{G}_{n+1}$ which proves (b).

Définition 5.1.5 *A Lie group G is a **nilpotent Lie group** if it is nilpotent as an abstract group.*

5.2 Nilpotent Lie Algebras

Let G be a Lie group with Lie algebra $\mathcal{G}$. We shall now define the notion of nilpotent Lie algebra so that if G is connected, then G is a nilpotent Lie group if and only if $\mathcal{G}$ is a nilpotent Lie algebra.

Définition 5.2.1 *(a) Let $\mathcal{G}$ be an algebra over a field $\Bbbk$ and let*

$C^0\mathcal{G} = \mathcal{G}\,;\, C^{n+1}\mathcal{G} = [\mathcal{G}, C^n\mathcal{G}]$. *Thus we see that*

$C^1\mathcal{G} = [\mathcal{G}, \mathcal{G}]$, ..., $C^k\mathcal{G} = (ad\mathcal{G})^k(\mathcal{G})$, ...

are ideals of $\mathcal{G}$ and we obtain the ***descending central series*** $\mathcal{G} = C^0\mathcal{G} \rhd C^1\mathcal{G} \rhd ....$

(b) Set $C_0\mathcal{G} = \{0\}$ and $C_{n+1}\mathcal{G} = \pi^{-1}(Z(\mathcal{G}/C_n\mathcal{G})$, where by induction $C_n\mathcal{G} \lhd \mathcal{G}$ and $\pi : \mathcal{G} \longrightarrow \mathcal{G}/C_n\mathcal{G}$ is the Lie algebra homomorphism and $Z(\mathcal{G}/C_n\mathcal{G})$ the center of the Lie algebra $\mathcal{G}/C_n\mathcal{G}$. Thus we see that $C_0\mathcal{G} = \{0\}$, $C_1\mathcal{G} = Z(\mathcal{G})$, etc. are ideals of $\mathcal{G}$ and we obtain the ***ascending central series*** $\{0\} = C_0\mathcal{G} \lhd C_1\mathcal{G} \lhd ....$

Théorème 5.2.2 *Let $\mathcal{G}$ be a Lie algebra over $\Bbbk$. Then the followind are equivalent.*

(a) There exists a sequence $\mathcal{G} = \mathcal{G}_0 \rhd \mathcal{G}_1 \rhd ... \rhd \mathcal{G}_s = \{0\}$ where all the $\mathcal{G}_n$ are ideals of $\mathcal{G}$ such that $[\mathcal{G}, \mathcal{G}_n] = \mathcal{G}_{n+1}$.

(b) There exists a positive integer p such that

$\mathcal{G} = C^0\mathcal{G} \rhd C^1\mathcal{G} \rhd ... \rhd C^p\mathcal{G} = \{0\}$.

(c) There exists a positive integer q such that $\{0\} = C_0\mathcal{G} \lhd C_1\mathcal{G} \lhd ... \lhd C_q\mathcal{G} = \lhd\, \mathcal{G}$.

(d) There rexists an integer r such that for all $X_1, X_2, ..., X_r \in \mathcal{G}$ *we have* $adX_1 \circ adX_2 \circ ... \circ adX_r = 0$.

PROOF The equivalence (a)-(c) are similar to Theorem 11.2. To see that (b) if and only (d) use the fact that $C^k\mathcal{G}$ is generated by the elements $(adX_1 \circ adX_2 \circ ... \circ adX_k)Y$ for any $X_1, X_2, ..., X_r$ and $Y \in \mathcal{G}$; see definition 11.6.

Définition 5.2.3 *A Lie algebra $\mathcal{G}$ is called **nilpotent** if it satifies any one of the conditions of theorem 11.7*

Remarks **(1)** Subalgebras, quotient algebras and finite direct sums of nilpotent Le algebras are also nilpotent.

(2) A nilpotent Lie algebra is solvable, for by induction we obtain

$C^n\mathcal{G} \subset \mathcal{G}^{(n+1)}$.

Proposition 5.2.4 *Let $\mathcal{G}$ be a Lie algebra over $\Bbbk$. $\mathcal{G}$ is solvable if and only if $[\mathcal{G},\mathcal{G}]$ is nilpotent.*

PROOF Suppose $[\mathcal{G},\mathcal{G}]$ is nilpotent, then $[\mathcal{G},\mathcal{G}]$ is solvable. Also since $\mathcal{G}/[\mathcal{G},\mathcal{G}]$ is a commutative Lie algebra, it is solvablle. Thus by corollary 10.8, $\mathcal{G}$ is solvable.

Conversly, first let P be the algebraic closure of $\Bbbk$ and let $\mathcal{G}$ be a solvable Lie algebra over P contained in $gl(V,P)$ where V is a finite-dimensional vector space over P. Then by the results following Lie's theorem (Proposition 10.23) there exists a basis of V so that =the matrices of $\mathcal{G}$ have triangular form

$$\begin{bmatrix} a_{11} & & & * \\ & a_{22} & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & a_{mm} \end{bmatrix}$$

and consequently the matrices for elements of $[\mathcal{G},\mathcal{G}]$ have the form

$$\begin{bmatrix} 0 & & & * \\ & 0 & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & 0 \end{bmatrix}$$

Thus by exercice (2), $[\mathcal{G},\mathcal{G}]$ is a nilpotent Lie algebra.

Next if $\mathcal{G}$ is an arbritrary Lie algebra over P, then $ad(\mathcal{G})$ is solvable, and therefore $[ad(\mathcal{G}),]\, ad(\mathcal{G}) = ad([\mathcal{G},\mathcal{G}])$ is nilpotent. However since

$ad : \mathcal{G} \longrightarrow ad(\mathcal{G})$ is a Lie algebra homomorphism with $Ker(ad) = Z(\mathcal{G})$ we see that $\overline{\mathcal{G}} = \mathcal{G}/Z(\mathcal{G}) \approxeq ad(\mathcal{G})$. Therefore $\overline{\mathcal{G}}^{(2)} = \left[\overline{\mathcal{G}},\overline{\mathcal{G}}\right] \approxeq ad([\mathcal{G},\mathcal{G}])$ is nilpotent. Consequently, there exists a positive integer p such that

$\{\overline{0}\} = C^{p+1}\overline{\mathcal{G}}^{(2)} = C^{p+1}\mathcal{G}^{(2)}/Z(\mathcal{G})$. Thus $C^{p+1}\mathcal{G}^{(2)} \subset Z(\mathcal{G})$ so that $C^{p+1}\mathcal{G}^{(2)} = \{0\}$; that is, $\mathcal{G}^{(2)} = [\mathcal{G},\mathcal{G}]$ is nilpotent.

Finally, if $\mathcal{G}$ is a Lie algebra over $\Bbbk$, we let $h = P \otimes_{\Bbbk} \mathcal{G}$ be the tensor product of the algebras P and $\mathcal{G}$ over $\Bbbk$ as in 9.1. Then h is a Lie algebra over P and a straightforward computation shows that $\mathcal{G}$ is solvable, then h is solvable. Thus since $[\mathcal{G},\mathcal{G}] \subset [h,h]$ we use the results of the preceding paragraph to conclude $[\mathcal{G},\mathcal{G}]$ is nilpotent.

Théorème 5.2.5 *(Engel's theorem) Let V be a nonzero finite-dimensional vector space over the field $\Bbbk$ and le $\mathcal{G}$ be a Lie subalgebra of $gl(V)$ which consists of nilpotent linear transformations (that is, $A^n = 0$ for some n). Then there is a nonzero vector $X \in V$ such that for all $A \in \mathcal{G}$, we have $AX = 0$.*

PROOF First we shall show that A being a linear nilpotent transformation implies that $ad_{\mathcal{G}}A$ is a nilpotent linear transformation acting on $\mathcal{G}$. Thus since $gl(V) = End(V)$ as sets, we can define the endomorphisms :

$$R(A) : \begin{array}{l} gl(V) \longrightarrow gl(V) \\ Z \longmapsto ZA \end{array} \quad \text{and} \quad L(A) : \begin{array}{l} gl(V) \longrightarrow gl(V) \\ Z \longmapsto AZ \end{array}$$

and see $(adA)Z = AZ - ZA = (L(A) - R(A))Z$ in $gl(V)$. Also noting that $L(A)R(A) = R(A)L(A)$ we have by the binomial theorem of any integer $k \geq 0$, $(adA)^k Z = [L(A) - R(A)]^k\, Z = \sum_{i=0}^{k} (-1)^i \binom{k}{i} A^{k-i} Z A^i$. However, since $A \in \mathcal{G}$ is nilpotent, all the factors A^{k-i} or A^i are 0 for suitably larger k or i. Thus $ad_{\mathcal{G}}A$ is nilpotent.

Next we shall use induction on $m = \dim \mathcal{G}$ to prove the result. For $m = 1$ we have $\mathcal{G} = \Bbbk A$ where A is a nonzero nilpotent linear transformation. Thus since there exists $X \in V$ with $AX = 0$, the same results holds for every $B = bA \in \mathcal{G}$. Now assume as an induction hypothesis that the result holds for Lie algebras of dimension less than m and let h be a proper subalgebra of $\mathcal{G}$ of maximum dimension. Then by the results of the above paragraph $ad_{\mathcal{G}} A$ is a nilpotent endomorphism on $\mathcal{G}$ for all $A \in h$. Thus since $adA : h \longrightarrow h$ we see adA induces a nilpotent endomorphism $\overline{A}$ on the vector space $\overline{\mathcal{G}} = \mathcal{G}/h$. Furtheremore the set $\overline{h} = \left\{\overline{A} : A \in h\right\}$ is a subalgebra of $gl(\overline{\mathcal{G}})$ which consists of nilpotent endomorphisms and $\dim \overline{h} < m$.

By the induction hypothesis with $V = \overline{\mathcal{G}}$ we can conclude that there exists $\overline{B} \neq \{\overline{0}\}$ in $\overline{\mathcal{G}}$ such that for all $\overline{A} \in \overline{h}$ we have $\overline{AB} = \overline{0}$; that is, there exists $B \in \mathcal{G}$ with $B \notin h$ and $[h, B] \subset h$. Thus the subspace $h + \Bbbk B$ of $\mathcal{G}$ is a subalgebra which contains h. However, by the maximal choice of h we have $h + \Bbbk B = \mathcal{G}$.

Finally let $W = \{Z \in V : AZ = 0, \text{ for all } A \in h\}$. Then by the above induction hypothesis $W \neq \{0\}$. Furtheremore for $A \in h$ and $B \in \mathcal{G}$ as above we have, since $[A, B] \in h$, that for any $Z \in W$

$$A(BZ) = (AB)Z = [A, B]\, Z + (BA)Z.$$

Thus by the definition of W we obtain $BW \subset W$. However, since B is nilpotent on V we have B is nilpotent on W. Consequently there exists $0 \neq X \in W$ with $BX = 0$ and since $\mathcal{G} = h + \Bbbk B$ we see this X has the desired property.

Corollaire 5.2.6 *Let V be a finite dimensional vector space over $\Bbbk$ and $\mathcal{G} \subset gl(V)$ be a Lie algebra of nilpotents endomorphisms of V.*

(a) There exists a basis of V such that the matices of the endomorphisms in $\mathcal{G}$ relative to this basis have the form

$$\begin{bmatrix} 0 & & & * \\ & 0 & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & 0 \end{bmatrix}$$

(b) $\mathcal{G}$ is a nilpotent Lie algebra of endomorphisms.

(c) The associative algebra $\mathcal{G}^*$ generated by the endomorphisms of $\mathcal{G}$ is a nilpotent associative algebra ; that is there is an integer r such that for any endomorphisms, $A_1, A_2, ..., A_r \in \mathcal{G}^*$ we have $A_1 A_2 ... A_r = 0$.

PROOF (a) Let $X_1 \in V$ such that $AX_1 = 0$ for all $A \in \mathcal{G}$. If the subspace $V_1 = \mathbb{k}X_1 \neq V$, then each $A \in \mathcal{G}$ induces a nilpotent endomorphism $\overline{A}$ on the nonzero vector space $\overline{V} = V/V_1$. Thus we can find $\overline{X_2} = X_2 + V_1 \neq \overline{0}$ in $\overline{V}$ such that $\overline{A}\ \overline{X_2} = \overline{0}$ for all $A \in \mathcal{G}$; that is, there exists $X_2 \in V$ and $X_2 \notin V_1$ with $AX_2 = a_{21}(A)X_1 + 0X_2$ for all $A \in \mathcal{G}$. Continuing we obtain a basis $X_1, X_2, ..., X_n$ of V such that for all $A \in \mathcal{G}$,

$AX_1 = 0$ and $AX_n \equiv 0\ mod(X_1, X_2, ..., X_{n-1})$ where $(X_1, X_2, ..., X_{n-1})$ denotes the subspace spanned by these vectors. Thus the matrix of A has $0'$s on and below the diagonal.

Part(b) follows from (a) and exercise 2, while (c) follows from (a) and matrix multiplication.

Corollaire 5.2.7 *Let $\mathcal{G}$ be an abstract Lie algebra over $\mathbb{k}$. Then $\mathcal{G}$ is a nilpotent Lie algebra if and only if for $X \in \mathcal{G}$ $ad_{\mathcal{G}}X$ is a nilpotent endomorphism of $\mathcal{G}$.*

PROOF If $\mathcal{G}$ is nilpotent, them from theorem 11.7(d) we have adX is nilpotent. Conversly, if each adX is nilpotent, then by corollary 11.11, $(ad\mathcal{G})^*$ is a nilpotent associative algebra. Thus there exists a positive integer p with $\{0\} = (ad\mathcal{G})^p = C^p\mathcal{G}$; that is, $\mathcal{G}$ is nilpotent.

Théorème 5.2.8 *Let G be a connected real Lie group with Lie algebra $\mathcal{G}$. Then G is a nilpotent Lie group if and only if $\mathcal{G}$ is a nilpotent Lie algebra.*

PROOF Thist assume that G is nilpotent and let

$G = G_0 \rhd G_1 \rhd ... \rhd G_n = \{e\}$ *a series of closed normal Lie subgroups of G such that* $(G, G_n) \subset G_{n+1}$. Consequently we have the corresponding series $\mathcal{G} = \mathcal{G}_0 \rhd \mathcal{G}_1 \rhd ... \rhd \mathcal{G}_n = \{0\}$ *of ideals of $\mathcal{G}$. Next* $(G, G_k) \subset G_{k+1}$ implies $[\mathcal{G}, \mathcal{G}_k] \subset \mathcal{G}_{k+1}$, for let $X \in \mathcal{G}$, $Y \in \mathcal{G}_k$. Then for t near 0 in $\mathbb{R}$ we have from theorem 5.16(c) that

$$(\exp(tX), \exp(tY) = \exp(t^2\,[X,Y] + o(t^3)) \text{ is in } G_{k+1}.$$

However from the charactezation of the Lie algebra of G_{k+1} from the theorem 6.9, this implies $t^2\,[X,Y] + o(t^3) \in \mathcal{G}_{k+1}$; that is, $[\mathcal{G}, \mathcal{G}_k] \subset \mathcal{G}_{k+1}$.

We now sketch the main parts of the proof of the converse and leave the details as exercises. First, since $\mathcal{G}$ is a nilpotent Lie algebra we see that $ad\mathcal{G}$ is a nilpotent Lie algebra of endomorphisms (with index of nilpotency N). Thus for any$Z \in \mathcal{G}$, adZ is nilpotent. Consequently in the expansion of theorem 5.18 $\exp X.\exp Y = \exp F(X,Y)$ for X, Y in $\mathcal{G}$ near the origin $0 \in \mathcal{G}$, we have that the Campbell-Hausdorff formula

$F(X,Y) = X + Y + \frac{1}{2}\,[X,Y] + ...$ is of finite lenght since

$(adX)^N = (adY)^N = 0.$

Secondly from the chain of ideals $\mathcal{G} = \mathcal{G}_0 \rhd \mathcal{G}_1 \rhd ... \rhd \mathcal{G}_n = \{0\}$, where $[\mathcal{G}, \mathcal{G}_k] \subset \mathcal{G}_{k+1}$, we obtain for the connected subgroup G_k generated by $\exp \mathcal{G}_k$ the chain $G = G_0 \rhd G_1 \rhd ... \rhd G_n = \{e\}$.

Finally, for $X \in \mathcal{G}$, $Y \in \mathcal{G}_k$ near enough the origin 0, we have for $x = \exp Y$, $y = \exp Y$: $xyx^{-1}y^{-1} = \exp([X,Y] + ...) = \exp P(X,Y)$ where $P(X,Y)$ is a finite sum of commutators, using the first part of the proof.Now each commutator term in $P(X,Y)$ contains $Y \in \mathcal{G}_k$. However $\mathcal{G}_k$ is an ideal of $\mathcal{G}$ so that $[\mathcal{G}, \mathcal{G}_k] \subset \mathcal{G}_{k+1}$ and therefore $P(X,Y) \in \mathcal{G}_{k+1}$. Thus

$xyx^{-1}y^{-1} \in \exp \mathcal{G}_{k+1} \subset G_{k+1}$ and by induction on the lenght of products of elements G and G_k we obtain $(G, G_k) \subset G_{k+1}$.

Exercice. Show that each commutatorterm in $P(X, Y)$ contains $Y \in \mathcal{G}_k$. Also complete the induction.

5.3 Nilpotent Lie Algebras of Endomorphisms

We shall now generalize the process of finding the Jordan canonical form matrix of an endomorphism to the process of decomposing a vector space into weight spaces relative to a nilpotent Lie algebra of endomorphisms ; that is, finding simultaneously "Jordan forms" for nilpotent Lie algebra of endomorphisms. Recall frol Section 9.2 that a Lie algebra over $\Bbbk$ is split if all the characteristic roots of adX are in $\Bbbk$ for all $X \in \mathcal{G}$.

Theorem 11.14 *Let V be a finite-dimensional vector space over $\Bbbk$ and let $\mathcal{G}$ be a split nilpotent Lie subalgebra of $gl(V)$.*

(a) There exists a direct sum decomposition $V = V(\phi_1) + ... + V(\phi_m)$, where $V(\phi_k) = \{X \in V : \text{for all } T \in \mathcal{G},\ (T - \phi_k(T)I)^p X = 0\}$ are $\mathcal{G}$-invariant weight spaces for $\mathcal{G}$ for $k = 1, ..., m$.

(b) There exists a basis of V so that the matrices of the endomorphisms in $\mathcal{G}$ relative to this basis all have the block form

$$\begin{bmatrix} \begin{bmatrix} \phi 1(T) & & & * \\ & . & & \\ & & . & \\ & & & . \\ 0 & & & \phi_1(T) \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} \phi_2(T) & & & * \\ & . & & \\ & & . & \\ & & & . \\ 0 & & & \phi_2(T) \end{bmatrix} \end{bmatrix}$$

(c) The functions $\phi_k : \mathcal{G} \longrightarrow \mathbb{k}$ are linear ; that is, $\phi_k \in \mathcal{G}^*$. Furtheremore $\phi_k([\mathcal{G}, \mathcal{G}]) = \{O\}$.

PROOF We break the proof into several parts. First we have the following formulla for an associative algenra A. Let s, $t \in A$, and let $s^{(0)} = s$, $s^{(1)} = ts - st = (adt)s$, and $s^{(k)} = (adt)^k s$. Then we obtain by induction for $k = 1, 2, ...$

$$\begin{aligned} t^k s &= \sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} s^{(i)} t^{k-i} \\ &= st^k + \begin{pmatrix} k \\ 1 \end{pmatrix} s^{(1)} t^{k-1} + ... + s^{(k)} \end{aligned} \qquad *$$

Next we have the following result.

Lemma 11.15 *Let V be a finite-dimensional vector space over $\mathbb{k}$, and let $\mathcal{G}$ be a split nilpotent Le subalgebra of $gl(V)$. Let $T \in \mathcal{G}$, $\lambda \in \mathbb{k}$ and let $V(\lambda) = \{X \in V : (T - \lambda I)^n X = 0$ for some $n\}$ be a weight space for T. Then $V(\lambda)$ is a $\mathcal{G}$-invariant subspace of V.*

PROOF We noted in section 10.3 that $V(\lambda)$ is a subspace. Since $\mathcal{G}$ is a nilpotent Lie algebra, the algebra $h = \mathcal{G} + \mathbb{k}I$ is also nilpotent where I is the identity endomorphism. Therefore by corollary 11.12, $[ad(T - \lambda I)]^N = 0$ for some fixed N. Now for $S \in \mathcal{G} \subset End(V)$ let $S^{(1)} = [ad(T - \lambda I)]\, S$, $S^{(2)} = [ad(T - \lambda I)]^2\, S$, etc. Then for $X \in V(\lambda)$ with $(T - \lambda I)^m X = 0$, we see by choosing $k = N + m$ and using $(*)$ that

$(T - \lambda I)^k (SX) = \sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} S^{(i)} (T - \lambda I)^{k-i} = 0$ noting $S^{(N)} = [ad(T - \lambda I)]^N S = 0$. Thus by definition of $V(\lambda)$ we see $SX \in V(\lambda)$; that is, $V(\lambda)$ is $\mathcal{G}$-invariant.

PROOF OF THEOREM 11.14 (continued) To prove part (a), we use induction on the dimension m of V. If $m = 1$, then every $T \in \mathcal{G}$ has a characteristic root so that $TX = \lambda(T)X$ for $V = \mathbb{k}X$. This yields the result in this case. For $m \gneqq 1$ we let $T \in \mathcal{G}$, and by Proposition 10.6 we have the direct sum $V = V(\lambda_1) + ... + V(\lambda_n)$ where $V(\lambda_i)$ are weight spaces for T. By Lemma 11.15 the $V(\lambda_i)$ are $\mathcal{G}$-invariant, and consequently $\mathcal{G}$ restricts to a nilpotent Lie algebra of endomorphisms on each $V(\lambda_i)$. Thus conclude the proof by induction, because we can assume T has at least two distinct characteristic roots (why ?) so that the dimension of $V(\lambda_i)$ is less than the dimension of V. Now since V is finite-dimensional, we see that there are only finitely many distinct weights ϕ_i of $\mathcal{G}$.

To construct the basis of V whitch gives the matrix in (b), it sufficies to find a basis for each $V(\phi_i)$ which gives the corresponding block matrix. Thus let $V(\phi)$ be a typical weight space as in (a). Then there is a nonzero $X \in V(\phi)$ such that $TX = \phi(T)X$. To see this, we use Lie's theorem (Theorem 10.23) replacing algebraic closure by "split" as follows. Since $\mathcal{G}$ is nilpotent on $V(\phi)$, it is solvable on $V(\phi)$. Thus there is a nonzero $X \in V(\phi)$ and a characteristic function F so that for all $T \in \mathcal{G}$, $TX = F(T)X$. Therefore $[F(T) - \phi(T)]\, X = [T - \phi(T)I]\, X$, and by induction $[F(T) - \phi(T)]^k\, X = [T - \phi(T)I]^k\, X = 0$ for k large enough, remembering $X \in V(\phi)$. Thus $F(T) = \phi(T)$; that si, $\phi(T)$ is the only characteristic root.

Thus, for $X_1 = X$ as above, the one-dimensional subspace $W = \Bbbk X_1$ is $\mathcal{G}$-invariant. Set $\overline{V} = V(\phi)/W$. Then $\mathcal{G}$ induces a nilpotent Lie algebra of endomorphisms $\overline{\mathcal{G}}$ by $\overline{T}\,\overline{X} = \overline{TX}$ From this, the characteristic roots of $\overline{T}$ are $\phi(T)$, and $\overline{V}$ is a weight space of dimension less than the dimension of V. By induction we can find a basis $\overline{X}_2, ..., \overline{X}_m$ of $\overline{V}$ so that

$$\begin{aligned}
\overline{T}\,\overline{X}_2 &= \phi(T)\overline{X}_2 \\
\overline{T}\,\overline{X}_3 &= a_{23}(\overline{T})\overline{X}_2 + \phi(T)\overline{X}_3 \\
&. \\
&. \\
&. \\
\overline{T}\,\overline{X}_m &= \sum_{j=2}^{m-1} a_{jm}(\overline{T})\overline{X}_j + \phi(T)\overline{X}_m.
\end{aligned}$$

Thus for $\overline{X}_i = X_i + W$ and $W = \Bbbk X_1$, we can find a basis $X_1, ..., X_m$ of

$V(\phi)$ so that

$$\begin{aligned} TX_1 &= \phi(T)\overline{X}_1 \\ TX2 &= a_{12}(T)X_1 + \phi(T)X_2 \\ &. \\ &. \\ &. \\ TX_m &= \sum_{j=1}^{m-1} a_{jm}(T)X_j + \phi(T)X_m. \end{aligned}$$

Thus we have the desired basis of $V(\phi)$.

For part (c), we show a weight ϕ is a linear funtional as follows. As in (b), let $0 \neq X \in V(\phi)$ be such that for all $T \in \mathcal{G}$, $TX = \phi(T)X$. Then for S, $T \in \mathcal{G}$ we have $S + T \in \mathcal{G}$ and

$\phi(S+T)X = (S+T)X = SX+TX = [\phi(S) + \phi(T)]\, X$ so that $\phi(S+T) = \phi(S) + \phi(T)$. Similarly, $\phi(aS) = a\phi(S)$ for $a \in \Bbbk$, and also $\phi([S,T])X = [S,T]\, X = (ST)X - (TS)X = \phi(S)\phi(T)X - \phi(T)\phi(S)X = 0$.

Thus since the elements of $[\mathcal{G},\mathcal{G}]$ are of the form $\sum [S,T]$, this implies $\phi([\mathcal{G},\mathcal{G}]) = 0$.

Chapitre 6

Topoligical groups

In our previous discussion of some matrix groups it was observed that we were studying not only the group operations but also the continuity of these operation. Thus in this chapter we abstract the situation and consider groups which are topological spaces so that the group operations are continuous relative to the topology of the space. We then prove facts for these topological groups which indicate that much information can be obtained from a neighborhood of the identity element ; this leads to local groups and local isomorphisms. Next we consider topological subgroups, coset spaces, and normal subgroups. Finally, for connected topological groups, we show that any neighborhood of the identity actually generates the group as an abstract group.

6.1 BASICS

In the next, we shall apply the results of the preceding chapters to obtain elementary results on Lie groups. However, since a Lie proup is a topological group, we shall briefty discuss this more general situation.

Définition 6.1.1 *A topological group is a set G such that :*

(a) *G is a Hausdorff topological space ;*

(b) *G is a group ;*

(c) the mappings $G \times G \longrightarrow G : (x, y) \longmapsto xy$ and $G \longrightarrow G : x \longmapsto x^{-1}$ are continuous, where $G \times G$ has the product topology.

Thus the set G has two structures-topological and algebraic-and they are related by property (c) that is, the group structure is compatible with the topological structure. The compatibility conditions in (c) are equivalent to the following single condition : (c') the mapping $G \times G \longrightarrow G : (x, y) \longmapsto xy^{-1}$ is continuous.

This condition holds for if (c) holds, then we have that $G \times G \longrightarrow G \times G : (x, y) \longmapsto (x, y^{-1})$ is continuous. Consequently the map $G \times G \longrightarrow G : (x, y) \longmapsto (x, y^{-1}) \longmapsto xy^{-1}$ is continuous. Conversely if (c') holds, then set $x = e$ (the identity) to obtain $(e, y) \longmapsto (e, y^{-1}) \longmapsto y^{-1}$ is a continuous map. Also from $xy = x(y^{-1})^{-1}$ the map $(x, y) \longmapsto xy$ is continuous. We can express (c) in terms of neighborhoods as follows. For any $x, y \in G$ and for any neighborhoods$W$ of xy in G, there exist neighborhoods U of x and V of y with $UV \subset W$. Also for any neighborhoods U of x^{-1}, we have if $U^{-1} = \{a^{-1} : a \in U\}$ is a neighborhoods of x. Thus replacing x by x^{-1}, we have if V is a neighborhoods of x, then V^{-1} is a neighborhoods of x^{-1}.

Définition 6.1.2 *Let G be a topological group and let $a \in G$. Then the map $L(a) : G \longrightarrow G : x \longmapsto ax$ is called a left tranlation. Similarty the map $R(a) : G \longrightarrow G : x \longmapsto xa$ is called a right translation.*

It should be noted that the maps $L(a)$ and $R(a)$ for $a \in G$ are homeomorphisms of G. Furthermore given any two points $x, y \in G$, then the homeomorphism $L(xy^{-1})$ maps x into y. In particular, there always exists a homeomorphism which maps $e \in G$ onto any other element $a \in G$ and using this, we shall see many of the local properties of $a \in G$ are determined by those of e. Thus, for example, U is a neighborhoods of $a \in G$ if and only if $U = L(a)V = aV$ where V is a neighborhoods of $e \in G$.

Proposition 6.1.3 *Let G be a topological space which is also a group. Then G is a topological group relative to these two structures if only if :*

(a) *the set* $\{e\}$ *is closed ;*

(b) *for all* $a \in G$ *the translations* $R(a)$ *and* $L(a)$ *are continuous ;*

(c) *the mappings* $G \times G \longrightarrow G : (x, y) \longmapsto xy^{-1}$ *is continuous at the point* (e, e).

6.2 Subgroups and Homogeneous Spaces

We shall consider subgroups H of a topological group G and the corresponding space of left cosets $G/H = \{aH : a \in G\}$. Then we eventually consider the case when H is a normal subgroup so that G/H becomes a topological group.

Définition 6.2.1 *Let* G *be a topological group and let* H *be a subset of* G *such that* $HH^{-1} \subset H$. *Then* H *is a subgroup of* G *(in the abstract sense). The topology of* G *induces a topology on the subgroup* H *by requiring* $U \subset H$ *to be open if only if* $U = H \cap V$ *where* V *is open in* G. *If with this induced topology, the subgroup* H *becomes a topological group we call* H *a topological subgroup.*

Remarque 6.2.2 *We shall be interested in closed subgroups* H *of* G *; that is,* H *is closed as a subset of* G. *However, we note that if* H *is an open subgroup of* G, *then* H *is closed. For, since* H *is open, so is* aH *for all* $a \in G$. *Therefore* $K = \cup\{aH : a \notin H\}$ *is open so that the complement of* K, *which is* H, *is closed.*

Théorème 6.2.3 *Let* H *be a topological subgroup of the topological group* G *and let* $\pi : G \longrightarrow G/H : a \longmapsto aH$ *be the natural projection. Then*

(a) G/H *can be made into a topological space such that :*

(i) *the projection* $\pi : G \longrightarrow G/H$ *is continuous, and*

(ii) *if* N *is a topological space and if* $f : G/N \longrightarrow N$ *is such that* $f \circ \pi : G \longrightarrow N$ *is continuous, then* f *is continuous.*

The topology defined on G/H *is uniquely determined by* (i) *and* (ii) *and is called the quotient topology.*

(*b*) *G/H with the quotient topology is such that π is an open map ; that is U is open in G implies $\pi(U)$ is open in G/H.*

(*c*) *The quotient topology is Hausdorff if and only if H is a closed subset of G.*

Définition 6.2.4 *A subgroup H is a normal subgroup of G if $aHa^{-1} \subset H$ all $a \in G$ and then G/H is a group relative to $aH \cdot bH = abH$ which is called the quotient ggroup.*

Corollaire 6.2.5 *Let H be a clesed normal subgroup of the topological group G and let G/H be the quotient group. Then relative to the quotient topology, G/H becomes a topological group such that the projection $\pi : G \longrightarrow G/H$ is an open continuous homomorphism.*

Preuve. It suffices to show that $G/H \times G/H \longrightarrow G/H : (aH, bH) \longmapsto ab^{-1}H$ is continuous. Let U be a neighborhood of $ab^{-1}H = \pi(ab^{-1})$ in G/H. Then $\pi^{-1}(U)$ is a neighborhood of ab^{-1} in G. Now there exist neighborhood V of a and W of b in G such that $VW^{-1} \subset \pi^{-1}(U)$. However, since π is open, $\pi(V)$ and $\pi(W)$ are neighborhoods of $aH = \pi(a)$ and $bH = \pi(b)$, respectively. Thus $\pi(V)\pi(W)^{-1} = \pi(VW^{-1}) \subset U$ which proves continuity. ■

Corollaire 6.2.6 *If H is an open normal subgroup of the topological group G, then G/H is discrete.*

Preuve. Since H is open, the cosets aH for $a \in G$ are open in G. Thus since π is an open map, the sets $\{aH\}$ in G/H are open. Therefore G/H is discrete. ■

Corollaire 6.2.7 *Let $f : G \longrightarrow \overline{G}$ be a homomorphism of topological groups. Then f is continuous if only if f is continuous at the identity $e \in G$.*

Preuve. Assume f is continuous at the identity. Let $a \in G$ and let $f(a)\overline{U}$ be a neighborhood of $f(a)$ in $\overline{G}$ where $\overline{U}$ is a neighborhood of $\overline{e}$ in $\overline{G}$. Since

f is continuous at $e \in G$ and since $\overline{e} = f(e)$, there exists a neighborhood U of e in G such that $f(U) \subset \overline{U}$ which proves continuity at a since aU is a neighborhood of a in G with $f(aU) \subset f(a)\overline{U}$. ■

Théorème 6.2.8 *Let $f : G \longrightarrow \overline{G}$ be a continuous homomorphism of the topological groups G and $\overline{G}$ and let $H = \{x \in G : f(x) = \overline{e}\}$ be the kernel of f where $\overline{e}$ is the identity of $\overline{G}$. Then : (a) H is a closed normal subgroup of G and $\pi : G \longrightarrow G/H$ is a continuous homomorphism;*

(b) there is a continuous monomorphism $g : G/H \longrightarrow \overline{G}$ such that $f = g \circ \pi$;

(c) let H and N be clased normal topological subgroups of G such that $N \subset H$. Then G/H is topologically isomorphic to $(G/N)\,(H/N)$.

Définition 6.2.9 *Let M be a Hausdorff topological space and let G be a topological group. Then :*

(a) G operates on M if there is a surjection $G \times M \longrightarrow M : (g, p) \longmapsto g \bullet p$ such that $(xy) \bullet p = x \bullet (y \bullet p)$ and $e \bullet p = p$ for all $x, y \in G$, and $p \in M$ where e is the identity of G.

(b) G operates transitively on M if for every $p, q \in M$, there exists $x \in G$ such that $x \bullet p = q$

(c) G operates continuously on M if the map $G \times M \longrightarrow M : (g, p) \longmapsto g \bullet p$ is continuous.

(d) G is called a topological transformation group on M if G operates continuously on M. Note that for each $x \in G$, the map $\tau(x) : M \longrightarrow M : p \longmapsto x \bullet p$ is a homeomorphism.

(e) G is effective if $a \bullet p = p$ for all $p \in M$ implies $a = e$.

(f) Let p be fixed in M. Then $G(p) = \{x \in G : x \bullet p = p\}$ is a group called the isotropy subgroup of G at p or fixed point subgroup at p. The set $G \bullet p = \{x \bullet p \in M : x \in G\}$ is called an orbit under G.

Théorème 6.2.10 *Let M be a Hausdorff space and let G be a transitive topological transformation group operating on M. Let p be some (fixed) point in M and let $G(p)$ be the isotropy group at p. Then $G(p)$ is a closed subgroup* *of G, and the map $f : G \longrightarrow G \bullet p : a \longmapsto a \bullet p$ induces a continuous bijection $\overline{f} : G/G(p) \longrightarrow M$ such that $\overline{f} \circ \pi = f$; that is, the accompanying diagram is commutative.*

Corollaire 6.2.11 *If $\overline{f} : G/G(p) \longrightarrow M$ is open or if $G/G(p)$ is compact, then $\overline{f}$ is a homeomorphism ; that is, M is a homogenous space.*

If G is a locally compact group with countable basic and if M is a locally compact Hausdorff space, then $\overline{f}$ is a homeomorphism of $G/G(p)$ onto M.

6.3 Connected Groups

Définition 6.3.1 *Let M be a topological space and let $p \in M$. Then p is contained in a unique maximal connected subset $C(p)$ is closed and is called the connected component of p. For $M = G$ a topological group, the connected component of the identity $e \in G$ is called the identity component of G and is denote by G_0.*

Théorème 6.3.2 *Let G be a topological group, and let G_0 be the identity component. Then :(a) G_0 is a closed normal topological subgroup of G and the connected component $C(a)$ of $a \in G$ equals aG_0;*

(b) If G is loccally connected (that is ; if every point $a \in G$ has a connected neighborhood), then G/G_0 is dixrete.

Proposition 6.3.3 *Let G be a topological group, let G_0 be the identity component, and let U be any open neighborhood of $e \in G$.*

(a) If U is a symmetric neighborhood, then $H =_{k=1}^{\infty} U^k$ is an open and closed subgroup of G. If U is connected, so is H. (b) $G_0 = \left(_{k=1}^{\infty} U^k\right) \cap G_0$. (c) If G is connected, then $_{k=1}^{\infty} U^k = G$. Thus any open neighborhood, of e is a set of generators of a connected topological group as an abstract group.

Définition 6.3.4 *Let G be a topological group. Then the center C of G equals $\{x \in G : xa = ax \text{ for all } a \in G\}$. The center is a normal subgroup of G and is also denote by $Z(G)$.*

Proposition 6.3.5 *Let G be a connected topological group and let H be a discrete normal topological subgroup of G. Then $H \subset G$, the center of G.*

Let G be a topological group and let H be a closed topological subgroup such that H is connected and G/H is connected. Then G is connected.

Définition 6.3.6 *A topological space M is locally Euclidean of dimension m if each point $p \in M$ has a neighborhood, which is homeomorphic to an open set in $\mathbb{R}^m$. Note that an open subset of $\mathbb{R}^m$ cannot be homeomorphic to an open subset of $\mathbb{R}^n$ if $m \neq n$.*

Chapitre 7

Haar measure

7.1 Integration on locally compact groups

The Haar measure was introduced for the first time in 1933 by A. Haar on a locally compact topological group, the Lebesgue measure $\mathbb{R}^n$.

7.1.1 Measure on locally compact space

Let X be a compact space.

Définition 7.1.1 *A measure on X is an element of the dual of the Banach space $C_c(X)$ of continuous complex functions in X, that is, a linear form $f \longrightarrow \mu(f)$ on $C_c(X)$ such that : $|\mu(f)| \leq a\,\|f\|$ for all $f \in \mathcal{C}_c(X)$ where*

$$\|f\| = \sup_{x \in X} |f(x)|$$

Définition 7.1.2 *Let X be a locally compact space (metrizable and separable). For any compact subset K of X, let $\mathcal{K}$ (X, K) be the vector subspace of $\mathcal{C}_c(X)$ formed by the support functions contained in K (hence compact). We put : $\mathcal{K}(X) = U\{\mathcal{K}(X, K),\ K \text{ compact de } X\}$. $\mathcal{K}$(X) is the vector space of continuous complex functions with compact support. A measure of Radon over*

X is a linear form μ over $\mathcal{K}$ (X) with the following property :For any compact subset K of X, there exists a number $M_K \geq 0$ such that for every function f$\in \mathcal{K}$ (X, K), we have. $|\mu(f)| \leq M_K \|f\|_\infty$*. This definition coincides with the previous one when X is compact. It expresses that the restriction* $\mu|_{\mathcal{K}(X,K)}$ *is continuous for the topology induced by that of* $\mathcal{C}_c^\infty(X)$*. We note that $\mathcal{K}$ (X, K) is closed in* $\mathcal{C}_c^\infty(X)$*, hence a Banach space. The value of the measure μ at the point* $f \in K(X,K)$ *is denoted. μ(f) or* $\int_X f(x)\, d\mu(x)$ *and is called the integral of f with respect to μ. We denote by* $M(X)$ *the set of measures of Radon over X. A measure μ is said to be real if μ(f) is real when f is real. The measure μ defined by : (f) =μ $(\overline{f})$ for all* $f \in K(X)$.

Définition 7.1.3 *a) Let X be a locally compact space and x a point of X.*
The mapping $f \longrightarrow f(x)$ *of* $\mathcal{K}(X)$ *into* $\mathbb{C}$ *is a measure because it is linear and we have for every compact subset K of X such that* $f \in \mathcal{K}(X,K)$, $|f(x)| \leq \|f\|$.

It is called the Dirac measure at point x (or the measure defined by the unit mass at point x and it is denoted by ε_x*.*

b) Let $f \in \mathcal{K}(\mathbb{R})$*. For all* $[a,b]$ *containing the support of f, we have :*

$$\int_a^b f(t)\, dt = \int_{-\infty}^{+\infty} f(t)\, dt.$$

The mapping $f \longrightarrow \int_{-\infty}^{+\infty} f(t)\, dt$ *is a linear form on* $\mathcal{K}(\mathbb{R})$*. Let us show that it is a measure. For any compact interval* $K = [a,b]$ *of* $\mathbb{R}$ *we have,* $\forall f \in \mathcal{K}(\mathbb{R},K)$;$\left|\int_{-\infty}^{+\infty} f(t)\, dt\right| \leq (b-a)\|f\|$ *. This measure is called the Lebesgue measure on* $\mathbb{R}$*.*

c) Let $\mu \in M(X)$, $g \in \mathcal{C}(X)$.

$$\forall f \in \mathcal{K}(\mathbb{R},K),\ g\,f \in \mathcal{K}(X)$$

and the mapping $f \longrightarrow \mu(gf)$ *is a linear form on* $\mathcal{K}(X)$*. Let us show that it is a measure.*

$$\forall f \in \mathcal{K}(\mathbb{R},K),\ \ \|gf\| \leq \|f\| .\underset{x\in K}{Sup}|g(x)|$$

and

$$|\mu(gf)| \leq b_K \|f\| \;\; où \; b_K = a_K \underset{x\in K}{Sup} |g(x)|$$

This measure is denoted g.μ and is called a density measure g with respect to μ. Let $\pi : X \longrightarrow X'$ *a homeomorphism of X onto a locally compact space X ':*

$$\forall f \in \mathcal{K}(X'),\; f \circ \pi \in \mathcal{K}(X)$$

et l'on a

$$Sup(f \circ \pi) = \pi^{-1}(Supp(f)).$$

We conclude that for any measure μ on X, $f \longrightarrow \mu(f \circ \pi)$ *is a measure on X ', called image of μ by π and denoted by π(μ).*

Let Y be a closed subset of X (hence a locally compact subspace) and 3bd a measure on Y.
$\forall f \in \mathcal{K}(X,K)$, the restriction $f|_Y \in \mathcal{K}(Y, K\cap Y)$, then, there exist a constant C_K such that

$$|v(f|_Y)| \leq C_K.Sup\,|f(y)| \leq C_K \|f\|,\; \forall f \in \mathcal{K}(X,K).$$

The mapping $f \longrightarrow \nu(f|_Y)$ is therefore a measure on X, called the image of γ by the canonical injection $Y \longleftrightarrow X$ (a canonical extension of ν to X).

We call the support of the measure μ, and Supp (μ) denotes the complement of the largest open μ-negligible in X.

$x \in Supp(\mu)$ means that for every function $f \in \mathcal{K}(X)$ such that $f(x) \neq 0$, we have $|\mu|(|f|) > 0$ or for every neighbourhood V of x, there exist $f \in \mathcal{K}(X)$, whose the support is contained in V and such that $\mu(f) \neq 0$. When $Supp(\mu) = X$, the only countinuous function μ- negligible is the constant 0.
By definition we have $Supp(\mu) = Supp(|\mu|)$ and it is clear that for any scalar $a \neq 0$, we have $Supp(a\,\mu) = Supp(\mu)$. Generally, for every function g locally $|\mu|$- integrable, we have $Supp(g.\mu) \subset Supp(g) \cap Supp(\mu)$, because if we put $\gamma = g.|\mu|$ and if an open set U does not match $Supp(g)$, or does not

match $Supp\,(\mu)$, we have $|\gamma|^*\,(U) = 0$.

7.1.2 Haar measure on topological group

Définition 7.1.4 *Let G be a locally compact group (metrizable and separable). For every complex function f defined on G, we put :*

$$_sf\,(x) = f\left(s^{-1}x\right)\ ;\ \ f_s\,(x) = f\,(x\,s)$$

(The Left and right translations of f by s), $\check{f}\,(x) = f\,(x^{-1})$ *and*

$$\tilde{f}\,(x) = \overline{f\,(x^{-1})}$$

For all $s, x \in G$.
it follows that :

$$_{st}f =_s ({}_tf) \quad et \quad f_{s\,t} = (f_s)_t\,.$$

The maps $x \longmapsto x^{-1}x$ and $x \longmapsto x\,s$ are homeomorphisms of G. Therefore functions $_sf$ and f_s are continuous if and only if f is continuous. In particular if $f \in \mathcal{K}\,(G)\,,\ {}_sf$ and f_s are also in $\mathcal{K}\,(G)$.
Let μ be a radon measure ; let $_s\mu$ and μ_s denote the measures on G, images of μ by homeomorphisms $x \longmapsto s\,x$ and $x \longmapsto xs^{-1}$ respectively. Hence we have

$$_s\mu\,(f) = \mu\,({}_{s^{-1}}f)\ \ et\ \mu_s\,(f) = \mu\,(f_{s^{-1}})\,,\ \forall f \in \mathcal{K}\,(G)$$

We say that μ is left invariant (resp. right) if, for all $s \in G$, we have $_s\mu = \mu$ (resp $\mu_s = \mu$) ie $\int_G f\,(s^{-1}x)\,d\mu\,(x) = \int_G f\,(x)\,d\mu\,(x)$ $(resp.\ \int f\,(x\,s)\,d\mu\,(x) = \int f\,(x)\,d\mu\,(x))$ .If a measure $\mu \neq 0$ on G is left invariant, we have $Supp\,(\mu) = G$ because

$$Supp\,({}_s\mu) = s\ Supp\,(\mu)\ \text{ for all } s \in G \text{ and } Supp\,(\mu) \neq \phi.$$

Similarly for the right invariant measures.

Définition 7.1.5 *Let G be a locally compact group. A left Haar measure (resp. right) on G is a left (resp. right) non-zero positive measure on G ;*
The existence of Haar measure is given by the following theorem :

Théorème 7.1.6 *There exists a left (resp. right) Haar measure μ, on every locally compact group G, and every left Haar measure (resp. right) on G is of the form $C\mu$ where C is a non-zero positive real number.*

Examples

a) The lebesgue measure dx on the additive group $\mathbb{R}^n \, (n \geq 1)$, is a Left and right Haar measure.

b) The measure $f \longrightarrow \int_0^{+\infty} \frac{f(t)}{t} dt$ is a left Haar measure on R_+^*.

Définition 7.1.7 *Let G be a locally compact group and μ a left Haar measure on G. For all $f \in \mathcal{K}(G)$ and all $s \in G$, the measure ν defined by :*

$$\nu(f) = \mu(f_{s^{-1}})$$

is a left invariant positive measure. Therefore, There exist an only one positive number $\triangle_G(s)$ such that $\nu(f) = \triangle_G(s)\,\mu(f)$ that is

$$\int_G f\left(x\, s^{-1}\right) \, d\mu(x) = \triangle_G(s) \int_G f(x) \, d\mu(\lambda).$$

The function $s \longmapsto \triangle_G(s)$ from G to R_+^* is called module function on G. The group G is said to be unimodular if $\triangle_G(s) = 1$ for all $p \in G$.

Remark :

If G is unimodular, Each left invariant measure on G is also right invariant. We say that this is a Haar measure on G.

Theorem

The map $s \longrightarrow \triangle_G(s)$ is a continuous homomorphism from G to the multiplicative group R_+^* of real positive numbers.

Proof :

Let μ be a left Haar measure on G and $f \in \mathcal{K}(G)$ such that $\mu(f) \neq 0$

$\forall x, y \in G$ we have :

$$\triangle_G(xy)\ \mu(f) = \mu\left(f_{(xy)^{-1}}\right) = \mu\left(\left(f_{y^{-1}}\right)_{x^{-1}}\right) = \triangle(x)\ \mu\left(f_{y^{-1}}\right) = \triangle(x)\,\triangle(y)\ \mu(f).$$

Hence

$$\triangle(xy) = \triangle(x)\ \triangle(y).$$

Let show that $\triangle_G$ is continuous. Given that $\triangle$ is a group homomorphism, we only have to show that $\triangle$ is continuous at point e.

Let $f \in \mathcal{K}(G)$ such that $\mu(f) = 1$ and let $\mathcal{K}$ be the compact support of f. Let U be a compact symmetric neighbourhood of e and put $S = KU$. S is compact, since K and U are also compact.

If $x \in U$, $f_{x^{-1}} - f$ is null in complementary S.

Indeed, if $f(yx^{-1}) \neq f(y)$, then $f(yx^{-1})$ and $f(y)$ can't be null together.

So $yx^{-1} \in K$ and $y \in Kx \subset KU = S$ or else $y \in K \subset S$.

A Haar measure is continuous, therefore there exist a constant $M \geq 0$ such that

$$\left\|\mu\left(f_{x^{-1}} - f\right)\right\| \leq M \left\|f_{x^{-1}} - f\right\|_\infty \qquad \forall x \in U.$$

As,

$$\begin{aligned}\mu\left(f_{x^{-1}} - f\right) &= \mu\left(f_{x^{-1}}\right) - \mu(f)\\ &= \triangle(x)\ \mu(f) - \mu(f)\\ &= (\triangle(x) - 1)\ \mu(f)\end{aligned}$$

we have

$$|\triangle(x) - 1| \leq M \left\|f_{x^{-1}} - f\right\|_\infty \qquad \forall x \in U$$

As every function $f \in \mathcal{K}(G)$ is uniformly continuous on G, then there exists a neighbourhood Ω of e, such that :

$$\left\|f_{x^{-1}} - f\right\|_\infty \leq \frac{\varepsilon}{M} \qquad \forall x \in \Omega$$

Putting $V = U \cap \Omega$, we have the result.

Théorème 7.1.8 *i) Commutatives or compacts topological groups are unimodular.*

ii) If H is a compact subgroup of a locally compact group G. Then $\triangle_H(\zeta) = \triangle_G(\zeta) = 1 \qquad \forall \zeta \in H$.

Remarque 7.1.9 *A subgroup of an unimodular group is not necessarily unimodular. For example $G = GL(2, \mathbb{R})$ is unimodular but the subgroup*

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \ a > 0 \ b \in \mathbb{R} \right\} \text{ de } G \text{ is not.}$$

Théorème 7.1.10 *Let G a locally compact group, $\triangle$ the module of G and μ a left Haar measure on G.*

$$\forall f \in \mathcal{K}(G) \qquad \int_G f\left(x^{-1}\right) \triangle \left(x^{-1}\right) d\mu(x) = \int_G f(x)\, d\mu(x)$$

Preuve. Put

$$v(f) = \int f\left(x^{-1}\right) \triangle \left(x^{-1}\right) d\mu(x)$$

γ is a linear form non identically null on $\mathcal{K}(G)$ and we have for all $s \in G$ and for all $f \in \mathcal{K}(G)$.

$$\begin{aligned} v({}_p f) &= \int_G f\left(p^{-1}x^{-1}\right) \triangle \left(x^{-1}\right) d\mu(x) \\ &= \int_G \check{f}(xp) \triangle \left(x^{-1}\right) d\mu(x) \\ & \triangle\left(p^{-1}\right) \int_G \check{f}(x) \triangle \left(px^{-1}\right) d\mu(x) \\ &= \int_G f\left(x^{-1}\right) \triangle \left(x^{-1}\right) d\mu(x) = \nu(f) \end{aligned}$$

Therefore ν is a left Haar measure. There exist a constant $a > 0$ such that $\check{\mu} = a\triangle^{-1}.\mu$;

It follows that $\mu = a\left(\triangle^{-1}.\mu\right) = a\triangle.\check{\mu} = a^2\mu$ Hence $a^2 = 1$, since $a > 0$, $a = 1$. ■

Corollaire 7.1.11 *Let μ a left Haar measure on a locally compact group G. Then G is unimodular if and only if*

$$\int_G f\left(x^{-1}\right)\, d\mu\,(x) = \int_G f\,(x)\; d\mu\,(x) \qquad\qquad *$$

for every function $f \in K\,(G)$.

Preuve. If G is unimodular, $\triangle_G\,(x) = 1$ for all $x \in G$.
Inversely if $\int_G f\,(x^{-1})\ d\mu\,(x) = \int_G f\,(x)\, d\mu\,(x)$ for all $f \in \mathcal{K}\,(G)$, G is unimodular since the first member is right invariant and the second member is left invariant by hypothesis. ■

Théorème 7.1.12 *For each function $f \in K\,(G)$, the map*

$$f \longmapsto \mu\,(f) \;=\; \int_G f\,(x)\; |J\,(L_x)|^{-1}\, dx$$

is a left Haar measure on G. Similarly the map

$$f \longmapsto \nu\,(f) \;-\; \int_G f\,(x)\; |J\,(R_x)|^{-1}\, dx$$

is a right Haar measure on G.

Exemple 7.1.13 *Let G be the set of square matrices of order 2 with real coefficients of the form*

$$g = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$$

where $x > 0$ and $y \in \mathbb{R}$.
G is a separately locally compact group and is isomorphic to the half-plane formed by $x \geq 0$. An element g of G can be written (x, y) with $(x, y)\ (u, v) = (xu, xv + y)$. If

$$s = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G,\ \textit{we have}$$

$$L_s\,(g) = s\,g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} . \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ax & ay + b \\ 0 & 1 \end{pmatrix}$$

$$R_s(g) = g\,s = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} . \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ax & bx+y \\ 0 & 1 \end{pmatrix}.$$

Hence

$$J(L_s) = a^2 \quad . \qquad J(R_s) = a.$$

As a function on G is identically a function of two variables x and y, then $f(g) = f(x,y)$. The left and right Haar measures on G can be written respectively, for all $f \in \mathcal{K}(G)$:

$$\int_G f(g)\; d\mu(g) = \int_{-\infty}^{+\infty}\int_0^{+\infty} \frac{f(x,y)}{x^2} dx\; dy$$

and

$$\int_G f(g)\; d\nu(g) = \int_{-\infty}^{+\infty}\int_0^{+\infty} \frac{f(x,y)}{x} dx\; dy.$$

We oberve that the group G is not unimodular.

Put $G = GL(2,\mathbb{R})$, if

$$s = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in G,$$

an elementary calculation show that

$$J(L_s) = J(R_s) = (a_{11}\,a_{22} - a_{12}\,a_{21})^2 = [det(s)]^2 .$$

Therefore the Left and Right Haar measures are the same and are given by

$$\int_G f(g)\; d\mu(g) = \int_{\mathbb{R}^4} \frac{f(x_{11},\, x_{12},\, x_{21},\, x_{22})}{x_{11}x_{22} - x_{12}x_{21}} dx_{11}\; dx_{12}\, dx_{21}\, dx_{22}.$$

Then $GL(2,\mathbb{R})$ is unimodular.

Let u be an automorphism of topological group G ; It is clear that the image $u^{-1}(\mu)$ of a left Haar measure μ on G is a left Haar measure again ; so there exists a number $a > 0$, which does not depend on the choice of μ, such that $u^{-1}(\mu) = a\,\mu$. We say that a is the module of automorphism u and let $mod_G(u)$ or $mod(u)$ denote it. For every function $f \in \mathcal{K}(G)$, we have :

$$\int_G f\left(u^{-1}(x)\right) d\mu(x) = mod(u) \int_G f(x)\, d\mu(x)$$

and in particular, for each set μ*-integrable* A, $\mu(u(A)) = mod(u)\ \mu(A)$.
For all $s \in G$, *Let* i_p *be the inner automorphism* $x \longrightarrow p^{-1}x\,p$; *We can write*

$$u_s^{-1}(\mu) = R_{(s)}\mu = \triangle_G(s)\,\mu.$$

Therefore $mod(i_s) = \triangle(s)$.
If G *is compact or discrete, we have* $mod(u) = 1$ *for all automorphism* u *of* G, *because* $u(G) = G$ *and* $u(\{e\}) = \{e\}$.
We have the following properties :

1) If u *and* v *are two automorphisms of* G *we have*

$$mod(u \circ v) = mod(u).mod(v).$$

2) For each automorphism u *of the vector space* $\mathbb{R}^n$, *we have :*

$$mod\,u = |\det(u)|$$

Theorem 5.3.1 *Let* G *a locally compact group,* P *and* Q *two closed subgroup of* G *such that* $G = PQ$. More precisely one assumes that the map

$$P \times Q \longrightarrow G,\ (x, y) \longmapsto xy$$

is a homeomorphism. Let $\triangle$ *denote the module of* G. *Let* α *denote a left Haar measure on* P *and* β *a right Haar measure on* Q. *Then the measure* μ defined on G *by*

$$\textstyle\int_G f(g)\mu(dg) = \int_{P\times Q} f(xy)\alpha(dx)\beta(dy)$$

is a left Haar measure on G.

Proof. For $g = xy$ $(g \in G, x \in P, y \in Q)$ let us write $x = \varphi_1(g)$,$y = \varphi_2(g)$.
Let μ be a left Haar measure on G. For $f_1 \in C_c(P)$, $f_2 \in Cc(Q)$, let us consider the integral

$$I(f_1, f_2) = \int_G f_1(\varphi_1(g))f_2(\varphi_2(g))\triangle(\varphi_2(g)^{-1})\mu(dg).$$

One can check that, for f_2 fixed, the map $f_1 \to I(f_1, f_2)$ defines a left invariant measure on P. Hence one can write

$$I(f_1, f_2) = B(f_2) \int_P f_1(x)\alpha(dx)$$

where B is a positive linear form on the space $C_c(Q)$ of continuous functions on Q with compact support. Similarly, for f_1 fixed, the map $f_2 \to I(f_1, f_2)$ defines a right invariant measure on Q, and therefore

$$I(f_1, f_2) = A(f_1) \int_Q f_2(y)\beta(dy)$$

where A is a positive linear form on $C_c(P)$. It follows that there is a positive constant C such that, for $f_1 \in Cc(P)$, $f_2 \in Cc(Q)$,

$$I(f_1, f_2) = C \int_{P\times Q} f_1(x)f_2(y)\alpha(dx)\beta(dy)$$

Therefore, if f is the function defined on G by

$$f(xy) = f_1(x)f_2(y),\ (x \in P,\ y \in Q),$$

then

$$\begin{aligned} \int_G f(g)\mu(dg) &= I(f_1, f_2\triangle) \\ &= C \int\int_{P\times Q} f_1(x)f_2(y)\triangle(y)\alpha(dx)\beta(dy) \end{aligned}$$

The statement is then proven for a function which can be written

$$f(g) = f_1(x)f_2(y)\triangle(y),\ (g = xy,\ x \in P,\ y \in Q),$$

where $f_1 \in C_c(P)$, $f_2 \in C_c(Q)$ and, by linearity, for a finite sum of such functions,

$$f(g) = \sum_{i=1}^{N} f_1^i(x)f_2^i(y)\triangle(y).$$

Since every function in $C_c(G)$ can be approximated by such functions for the topology of $C_c(G)$, the statement is now proven. □

Let us give a first application of this theorem. Let $G = GL(n, \mathbb{R})$, $K = O(n)$, and $T = T(n, \mathbb{R})+$ the group of upper triangular matrices with positive diagonal entries. Let us recall the Gram decomposition (Theorem 1.6.1). Every element g in G can be written

$$g = kt,$$

with $k \in K$, $t \in T$. The decomposition is unique, and the map

$$\varphi : K \times T \to G,\ (k, t) \to kt,$$

is a homeomorphism.

Proposition 5.3.2 *Let $K = O(n)$ and $T = T(n, \mathbb{R})+$, and let α denote the normalised Haar measure of K. There exists a constant $c_n > 0$ such that, for every function f, which is integrable on $G = GL(n, \mathbb{R})$,*

$$\int_G f(x)\,|\det(x)|^{-n} \prod_{i,j=1}^{n} dx_{ij} = c_n \int_{K\times T} f(kt)\alpha(dk) \prod_{i=1}^{n} t_{ii}^{-1} \prod_{i\leq j}^{n} dt_{ij}.$$

Proof. We saw that the group $G = GL(n, \mathbb{R})$ is unimodular (Example 3), and that the measure defined on T by

$$\beta_T(dt) = \prod_{i=1}^{n} t_{ii}^{-1} \prod_{i\leq j}^{n} dt_{ij},$$

is a right Haar measure (example 5). Hence this proposition is a direct consequence of Theorem 5.3.1. □

For the evaluation of the constant c_n, see Exercice 4.One can also consider the decomposition

$$\psi : T \times K \to G,\ (t, K) \to tk,$$

and, similarly, there exists a constant $d_n > 0$ such that

$$\int_G f(x)\,|\det(x)|^{-n} \prod_{i,j=1}^{n} dx_{ij} = d_n \int_{T\times K} f(tk) \prod_{i=1}^{n} t_{ii}^{i-n-1} \prod_{i\leq j}^{n} dt_{ij}\alpha(dk)$$

In fact,

$$\beta_l(dt) = \prod_{i=1}^{n} t_{ii}^{i-n-1} \prod_{i\leq j}^{n} dt_{ij}$$

is a left Haar measure on the group T. One can show that $c_n = d_n$ (see Exercise 4).

7.2 Some facts about differential calculus

We saw how it is possible to determine a Haar measure on a group Gwhich can be realised as an open set in $\mathbb{R}^m$, and when the transformations

$L(g)$ and $R(g)$ are restrictions to G of affine linear maps. This method does not apply to groups whose geometry is less simple, such as the

orthogonal group $O(n)$ or the unitary group $U(n)$. We will see in Section 5.5 how it is possible to determine a Haar measure on a linear Lie

group by using differential forms. For that we will first recall some facts in differential calculus.

Let V be a submanifold in $\mathbb{R}^N$, and x_0 a point of V. A tangent vector X at x_0 can be written

$$X = \gamma'(t_0),$$

where γ is a $\mathcal{C}^1$ curve drawn on V such that $\gamma(t_0) = x_0$. The tangent vectors at x_0 form a vector subspace in $\mathbb{R}^N$ which is called the tangent

vector space of V at x_0 and is denoted by $T_{x_0}(V)$. Let V and W be two submanifolds in $\mathbb{R}^N$, and φ a differential map from

V into W. The image under φ of a $\mathcal{C}^1$ curve γ which is drawn on Vrunning through x_0 is a curve $\varphi \circ \gamma$ which is drawn on W running through

$y_0 = \varphi(x_0)$, and

$$\frac{d}{dt}\varphi \circ \gamma(t) \mid_{t=t_0} = D\varphi_{x_0}(\gamma'(t_0))$$

If V and W have the same dimension and if φ is a diffeomorphism, then the differential $(D\varphi)_x$ of φ at every point $x \in V$ is an isomorphism from

$T_x(V)$ onto $T_y(W)$, where $y = \varphi(x)$. If V and W have the same dimension and if, for every $x \in V$, the differential $(D\varphi)_x$ is an isomorphism, then (V, φ) is a covering of W.

A vector field ξ on V is the prescription at each point x of V of a tangent vector $\xi(x)$ in $T_x(V)$. It is said to be $\mathcal{C}^k$ if $x \longmapsto \xi(x)$ is $\mathcal{C}^k$.

Let $\varphi : V \to W$ be a diffeomorphism from V onto W, and ξ a vector field on V. The image of ξ under φ is denoted by $\varphi * \xi$:

$$(\varphi * \xi)(\varphi(x) = D\varphi_x(\xi(x)).$$

To every vector field ξ on V one associates the differential operator $\widetilde{\xi}$ of order one defined by

$$\widetilde{\xi} f(x) = Df_x(\xi(x)).$$

If ξ is a vector field on V, then the map $f \longmapsto \widetilde{\xi} f$ is a derivation of the algebra $\mathcal{C}^\infty(V)$:

$$\widetilde{\xi}(fg) = \widetilde{\xi}(f)g + f\widetilde{\xi}(g),\ (f, g \in \mathcal{C}^\infty(V)),$$

and one can show that every derivation of $\mathcal{C}^\infty(V)$ is obtained in that way.The space of $\mathcal{C}^\infty(V)$ vector fields on V will be denoted by $\Xi(V)$.

If ξ and η are two vector fields in $\Xi(V)$ their bracket $[\xi, \eta]$ is defined by

$$\widetilde{[\xi, \eta]} = \widetilde{\xi} \circ \widetilde{\eta} - \widetilde{\eta} \circ \widetilde{\xi}.$$

Hence (V) is equipped with a Lie algebra structure. If the vector fields ξ and η are written in local coordinates

$$\xi(x) = (\xi_1(x), ..., \xi_n(x))\,;\, \eta(x) = (\eta_1(x), ..., \eta_n(x)),\ (m = dimV)$$

then

$$\widetilde{\xi}(f) = \sum_{i=1}^{m} \xi_i \frac{\partial f}{\partial x_i}\,; \quad \widetilde{\eta}(f) = \sum_{i=1}^{m} \eta_i \frac{\partial f}{\partial x_i},$$

and

$$\begin{aligned}\tilde{\xi} \circ \tilde{\eta} f - \tilde{\eta} \circ \tilde{\xi} f &= \sum_{i=1}^{m} \xi_i \frac{\partial}{\partial x_i} (\sum_{i=j}^{m} \eta_j \frac{\partial f}{\partial x_j}) - \sum_{i=1}^{m} \eta_i \frac{\partial}{\partial x_i} (\sum_{i=j}^{m} \xi_j \frac{\partial f}{\partial x_j}) \\ &= \sum_{i=1}^{m} \zeta_i \frac{\partial f}{\partial x_i}\end{aligned}$$

with

$$\zeta_i = \sum_{i=j}^{m} (\xi_j \frac{\partial \eta_i}{\partial x_j} - \eta_j \frac{\partial \xi_i}{\partial x_j}).$$

A *differential form* of degree one is a map $\alpha : \Xi(V) \to \mathcal{C}^{\infty}(V)$ which is $\mathcal{C}^{\infty}-$ linear :

$$\alpha(\xi f) = f\alpha(\xi), \quad (f \in \mathcal{C}^{\infty}).$$

The function $\alpha(\xi)$ can be written

$$x \longmapsto \alpha_x(\xi(x)), \text{ where } \alpha_x \text{ is a linear form on } T_x(V).$$

Let $u \in \mathcal{C}^{\infty}(V)$, then the map

$$\alpha : \xi \longmapsto Du(\xi)$$

defines a differential form. One writes $\alpha = du$. Let φ be a $\mathcal{C}^{\infty}(V)$ map from a manifold V into a manifold W. If α is a differential form of degree one on W, one denotes by $\varphi^*\alpha$ the differential form defined on V by

$$\varphi^*\alpha(\xi) = \alpha(D\varphi(\xi)).$$

In a system of local coordinates a differential form α of degree one can be written as a linear combination with coefficients in $\mathcal{C}^{\infty}(V)$ of the differential dx_i of the coordinates :

$$\alpha = \sum_{i=1}^{m} \alpha_i(x) dx_i.$$

A *differential form* of degree k on V is a map

$$\Xi(V) \times \cdots \times \Xi(V) \to \mathcal{C}^{\infty}(V)$$

which is $k - \mathcal{C}^\infty(V)-$linear and alternate. If ω is a differential form of degree k on V, and if $\xi_1, \ldots, \xi_k$ are k $\mathcal{C}^\infty$ vector fields on V, then

$\omega(\xi_1, ..., \xi_k)$ is a unction on V which can be written

$x \longmapsto \omega_x(\xi_1(x), ..., \xi_k(x))$ where ω_x is a $k-$skewlinear form on $T_x(V)$.

The wedge product $\alpha_1 \wedge \cdots \wedge \alpha_k$ of k linear forms $\alpha_1, ..., \alpha_k$ of degree one is the differential form of degree k defined by

$$\alpha_1 \wedge \cdots \wedge \alpha_k(\xi_1, ..., \xi_k) = det(\alpha_i(\xi_j))_{1 \leq i,j \leq k}.$$

If φ is a differential map from V into W, and if ω is a differential form of degree k on W, one denotes by $\varphi^*\omega$ the differential form of degree k

defined on V by

$$\varphi^*\omega(\xi_1, ..., \xi_k) = \omega(D\varphi(\xi_1), ..., D\varphi(\xi_k)).$$

If $X_1, ..., X_k$ are k tangent vectors at $x \in V$,

$$(\varphi^*\omega)_x(X_1, ..., X_k) = \omega_{\varphi(x)}((D\varphi)_x X_1, ..., (D\varphi)_x X_k).$$

This is an important formula that we will use several times in the following. A differential form ω of degree m on an open set V in $\mathbb{R}^m$ can be written

$$\omega = a(x)dx_1 \wedge \cdots \wedge dx_m,$$

where a is a function defined on V. Let φ be a diffeomorphism from Vonto W, where V and W are two open sets in $\mathbb{R}^m$, and ω a differential

form of degree m on W;

$$\omega = a(y)dy_1 \wedge \cdots \wedge dy_m$$

Then

$$\varphi^*\omega = a(\varphi(x))J\varphi(x)dx_1 \wedge \cdots \wedge dx_m,$$

where $J\varphi$ is the Jacobian determinant of φ,

$$J\varphi = \det(\frac{\partial \varphi_j}{\partial x_i})_{1 \leq i,j \leq m}$$

To every differential form ω of degree m on a manifold V of dimension $m$one associates a positive measure which is called the modulus of ω and

denoted by $|\omega|$. Let V_0 be an open set where local coordinates are available.

In V_0 the form ω can be written

$$\omega = a(x)dx_1\wedge\cdots\wedge dx_m,$$

and the measure $|\omega|$ has the density $|a(x)|$ with respect to the Lebesgue measure : $|\omega|\,(dx) = |a(x)|\,dx_1\cdots dx_m$

If φ is a diffeomorphism from V onto W, and if ω is a differential form of degree m on W, then

$|\varphi^*\omega| = \varphi^{-1}(|\omega|)$that is, if f is a continuous function with compact support on W,

$$\textstyle\int_W f(y)\,|\omega|\,(dy) = \int_V f\circ\varphi(x)\,|\varphi^*\omega|\,(dx).$$

In terms of local coordinates this relation is nothing but the change of variable formula for multiple integrals :

$$\textstyle\int_{W_0} f(y)\,|a(y)|\,dy_1...dy_m = \int_{V_0} f(\varphi(x))\,|a(\varphi(x))|\,|J\varphi(x)|\,dx_1...dx_m.$$

More generally, if φ is a covering of order k,

$$\textstyle\int_V f\circ\varphi(x)\,|\varphi^*\omega|\,(dx) = k\int_W f(y)\,|\omega|\,(dy).$$

If φ is a diffeomorphism from V onto V, and if ω is a differential form of degee m on V, which is invariant under φ up to a sign, that is $\varphi^*\omega = \pm\omega$,

then the measure $|\omega|$ is invariant under φ, that is, if φ is a continuous function on V with compact support,

$$\textstyle\int_V f\circ\varphi(x)\,|\omega|\,(dx) = \int_V f(x)\,|\omega|\,(dx)$$

7.3 Invariant vector fields and Haar mesure one a linear Lie groups

Let G be a linear Lie group, that is a closed subgroup in $GL(n,\mathbb{R})$. It is a submanifold in $M(n,\mathbb{R})$ (Corollary 3.3.5).

Proposition 5.5.1 *The tangent vector space to G at the identity element $e = I$ is the Lie algebra $\mathcal{G} = Lie(G)$ of G.*

Proof. (a) Let $X \in \mathcal{G}$. Then $\gamma(t) = exptX$ is a curve drawn on G running through e for $t = 0$, and $\gamma'(0) = X$, hence $X \in T_e(G)$ and $\mathcal{G} \subset T_e(G)$.

(b) Conversely let $\gamma(t)$ be a curve drawn on G running through e for $t = t_0$. For t close to t_0, $X(t) = log\gamma(t)$ is well defined and $t \longmapsto X(t)$ is a curve in g ; furthermore

$$\gamma'(t_0) = (D\exp)_0(X'(t_0)).$$

Since $(D\exp)_0 = Id$, $\gamma'(t_0) = X'(t_0) \in \mathcal{G}$. This shows that $T_e(G) \subset \mathcal{G}$. □

To $X \in \mathcal{G}$ one associates the vector field ξ_X on G defined by

$$\xi_X(g) = (DL(g))_e(X) = g.X.$$

This is a left invariant vector field : it is invariant under the diffeomorphisms $L(g) : x \longmapsto gx$,

$$L(g)_*\xi_X = \xi_X.$$

To this vector field one associates the left invariant differential operator

$$(\widetilde{\xi}_X f)(g) = (Df)_g(g.X) = \frac{d}{dt}\,|_{t=0}\, f(g\exp tX).$$

Proposition 5.5.2 *The map $X \longmapsto \xi_X$ is an isomorphism of the Lie algebra $\mathcal{G}$ onto the Lie algebra made of the left invariant vector fields on G.*

Proof. The map

$$\mathcal{G} \to \Xi(G), \quad X \longmapsto \xi_X \ ,$$

is injective, and every left invariant vector field on G is of that form. For $X, Y \in \mathcal{G}$,

$$(\left[\widetilde{\xi}_X, \widetilde{\xi}_Y\right] f)(g) = (Df)_g(g.[X,Y]) = (\widetilde{\xi}_{[X,Y]} f)(g)$$

In fact,

$$\begin{aligned}\widetilde{\xi}_X.\widetilde{\xi}_Y f(g) &= \tfrac{d}{ds}\,|_{s=0}\, \tfrac{d}{dt}\,|_{t=0}\, f(g\exp sX \exp tY)\\ &= (D^2f)_g(gX, gY) + (Df)_g(gXY).\end{aligned}$$

□

Let ω be a left invariant differential form of degree k on G. Then, if $X_1, ..., X_k \in \mathcal{G} = T_e(G)$,

$$\omega_g(gX_1, ..., gX_k) = \omega_e(X_1, ..., X_k).$$

Hence, the form ω is determined by ω_e, which is a k-skewlinear form on $\mathcal{G}$. Conversely, given a k-skewlinear form ω_0 on $\mathcal{G}$, there is a unique

left invariant differential form ω of degree k on G such that $\omega_e = \omega_0$.

Proposition 5.5.3 *Let ω be a (non-zero) left invariant differential form of degree $m = dimG$ on G. Then $|\omega|$ is a left Haar measure on G.*

Proof. In fact, if $\varphi = L(g)$, then $\varphi^*\omega = \omega$ and, for every continuous function f on G with compact support,

$$\int_G f(gx)\,|\omega|\,(dx) = \int_G f(x)\,|\omega|\,(dx)$$ □

In Section 5.2 we considered the case of a group G which can be identified with an open set in $\mathbb{R}^m$. This means that there exists on G a

system of global coordinates. We can rephrase what was said. Let ωbe a differential form on G of degree m :

$$\omega = a(x)dx_1\wedge\cdots\wedge dx_m.$$

If $J_g(x)$ denotes the Jacobian determinant of $L(g)$ at x,

$$\varphi^*\omega = a(g.x)J_g(x)dx_1\wedge\cdots\wedge dx_m.$$

The measure $|\omega|$ is left invariant if

$$a(g.x) = \pm a(x).$$

Therefore, the measure μ defined by

$$\mu(dx) = \frac{C}{|J_x(e)|}dx_1...dx_m$$

where C is a positive constant, is a left Haar measure.

Proposition 5.5.4

$$\triangle(g) = |\det Ad(g^{-1})|$$

Proof. Let ω be a left invariant differential form of degree m on G.

For $g \in G$, the inner automorphism

$x \longmapsto \varphi(x) = gxg^{-1}$ is a diffeomorphism of G. Let us show

that

$$\varphi^*\omega = \det Ad(g)\omega.$$

For $X_1, ..., X_m \in g$,

$$\begin{aligned}(\varphi^*\omega)_x(xX_1, ..., xX_m) &= \omega_{gxg_{-1}}(g(xX_1)g^{-1}, ..., g(xX_m)g^{-1}) \\ &= \omega_{gxg_{-1}}(gxg^{-1}Ad(g)X_1, ..., gxg^{-1}Ad(g)X_m) \\ &= \omega_e(Ad(g)X_1, ..., Ad(g)X_m) \\ &= \det Ad(g)\omega_e(X_1, ..., X_m) \\ &= \det Ad(g)\omega_x(xX_1, ..., xX_m)\end{aligned}$$

It follows that, if μ denotes the left Haar measure associated to ω, then

$$\int_G f(gxg^{-1})\mu(dx) = |\det Ad(g)|^{-1} \int_G f(x)\mu(dx)$$

and $\triangle(g) = |\det Ad(g)|^{-1}$. □

Corollary 5.5.5 *In the three following cases the group G is unimodular :*

(i) $Ad(G)$ *is compact,*
(ii) $\mathcal{G} = Lie(G)$ *is semi-simple,*
(iii) $\mathcal{G} = Lie(G)$ *is nilpotent and G is connected.*

We already saw that a compact group is unimodular (Proposition 5.1.3). In (iii) it is necessary to assume that G is connected. In fact $\mathcal{G}$ could be nilpotent and G non-unimodular. (See Exercise 3.).

Proof. (i) The map

$$G \to \mathbb{R}_+^*,\ g \longmapsto |\det Ad(g)|$$

is a continuous morphism. If $Ad(G)$ is compact, then this map is bounded, hence constant and equal to 1.

(ii) Let B be the Killing form of $\mathcal{G}$. From the relation

$$ad(Ad(g)) = Ad(g)adX\,Ad(g^{-1}), \quad (g \in G, X \in \mathcal{G}),$$

it follows that

$$B(Ad(g)X, Ad(g)Y) = B(X,Y), \quad (X, Y \in \mathcal{G}).$$

This means that $Ad(g)$ belongs to the orthogonal group of B. Therefore

$$|\det Ad(g)| = 1.$$

(iii) For every $X \in \mathcal{G}$, adX is nilpotent and

$\det Ad(expX) = \det ExpadX = e^{tr(adX)} = 1$. Therefore, for every g in a neighbourhood of e,

$$\det Ad(g) = 1.$$

Hence, the subgroup

$H = \{g \in G \mid \det Ad(g) = 1\}$is open and closed and, since G is connected, $H = G$. □

Let φ be the diffeomorphism of G defined by

$$x \longmapsto \varphi(x) = x^{-1}.$$

One can show that, if ω is a left invariant differential form of degree m, then

$$\varphi^*\omega = \det(-Adx)\omega.$$

(See Exercise 6.)

In the following proposition we express left Haar measures in the exponential chart.

Proposition 5.5.6 *Let U be a connected neighbourhood of 0 in $\mathcal{G} = Lie(G)$ such that the exponential map is a diffeomorphism of U onto $V = expU$. Let μ be a left Haar measure on G and λ a Lebesgue measure on $\mathcal{G}$. Let f be an integrable function on G supported in V. Then*

$$\int_G f(g)\mu(dg) = c\int_{\mathcal{G}} f(\exp X)\det A(X)\lambda(dX),$$

where

$$A(X) = \frac{I - \exp(-adX)}{adX}$$

and c is a positive constant.

Observe that, if $\mathcal{G}$ is nilpotent then, for every X, adX is nilpotent and $\det A(X) = 1$.

Proof. Let ω be a left invariant differential form of degree m on G. For $\varphi = exp$, the exponential map,

$$(\varphi^*\omega)_X(Y_1, ..., Y_m) = \omega_{\exp X}((D\exp)_X Y_1, ..., (D\exp)_X Y_m),$$

and, by Theorem 2.1.4,

$$\begin{aligned}(\varphi^*\omega)_X(Y_1, ..., Y_m) &= \omega_{\exp X}(\exp X A(X)Y_1, ..., \exp X A(X)Y_m)\\ &= \omega_e(A(X)Y_1, ..., A(X)Y_m)\\ &= \det A(X)\omega_e(Y_1, ..., Y_m).\end{aligned}$$

Since the exponential map is a diffeomorphism from U onto V it follows-that $\det A(X) = 0$. Therefore, since U is connected, and $A(0) = Id$, then $\det A(X) > 0$ on U. □

Chapitre 8

Representations of compact groups

In the mathematical field of representation theory, group representations describe abstract groups in terms of linear transformations of vector spaces ; in particular, they can be used to represent group elements as matrices so that the group operation can be represented by matrix multiplication. Representations of groups are important because they allow many group-theoretic problems to be reduced to problems in linear algebra, which is well understood. They are also important in physics because, for example, they describe how the symmetry group of a physical system affects the solutions of equations describing that system.

The term representation of a group is also used in a more general sense to mean any "description" of a group as a group of transformations of some mathematical object. More formally, a "representation" means a homomorphism from the group to the automorphism group of an object. If the object is a vector space we have a linear representation. Some people use realization for the general notion and reserve the term representation for the special case of linear representations. The bulk of this article describes linear representation theory ; see the last section for generalizations.

In this chapter we present the Peter–Weyl theory for compact groups. By using spectral theory for compact operators we will see that an irreducible representation of a compact group is finite dimensional. Using the Peter–Weyl theory, classical Fourier analysis is extended to compact groups.

8.1 Unitary representations

Let G be a topological group and V a normed vector space over $\mathbb{R}$ or $\mathbb{C}$ ($V \neq \{0\}$). Let (V) denote the algebra of bounded operators on V. A representation of G on V is a map

$$\begin{array}{c} \pi : G \longrightarrow (V) \\ g \longmapsto \pi(g) \end{array},$$

such that

1. $\pi(xy) = \pi(x)\pi(y)$, $\pi(e) = I$,
2. for every $v \in V$, the map $\begin{array}{c} \pi : G \longrightarrow V \\ g \longmapsto \pi(g)v \end{array}$, is continuous.

The definition, we give here, differs slightly from that given in Section, where we only considered the case of a finite dimensional vector space V. A subspace $W \subset V$ is said to be invariant if, for every $g \in G$, $\pi(g)W = W$. Putting $\pi_0(g) = \pi(g)W$, the restriction of $\pi(g)$ to W, we get a representation of G on W. One says that π_0 is a subrepresentation of π. Assume W to be closed. The representation π_1 of G on the quotient space V/W is called the quotient representation. The representation π is said to be irreducible if the only invariant closed subspaces are $\{0\}$ and V. Observe that, by definition, a one dimensional representation is irreducible. Let (π_1, V_1) and (π_2, V_2) be two representations of G. If a continuous linear map A from V_1 into V_2 satisfies the relation $A\pi_1(g) = \pi_2(g)A$, for every $g \in G$, one says that A is an intertwinning operator or that A intertwins the representations π_1 and π_2. The representations (π_1, V_1) and (π_2, V_2) are said to be equivalent if there exists an isomorphism $A : V_1 \longrightarrow V_2$ which intertwins the representations π_1 and π_2. Let H be a Hilbert space. Recall that an operator A onHis said to be unitary if it is invertible and $A^{-1} = A^*$. A representation π of G on H is said to be unitary if, for every $g \in G$, $\pi(g)$ is a unitary operator ; this can be written

$$\forall g \in G, \forall v \in H, \|\pi(g)v\| = \|v\| .$$

If the representation π is unitary, and if W is an invariant subspace, then the orthogonal subspace $W^{\perp}$ is invariant as well. If W is closed, the quotient representation on H/W is equivalent to the subrepresentation on $W^{\perp}$.

Proposition 8.1.1 *Let π be a representation of a compact group G on a finite dimensional vector space V. There exists on V a Euclidean inner product for which π is unitary.*

Preuve. Let us choose arbitrarily on V a Euclidean inner product $(\cdot|\cdot)_0$ and put

$$(u, v) = \int_G (\pi(g)u, \pi(g)v)_0 \, \mu(dg),$$

where μ is a Haar measure on G. One can check easily that $(\cdot|\cdot)$ is a Euclidean inner product on V, and that the representation π is unitary with respect to this Euclidean inner porduct. ■

Corollaire 8.1.2 *Let π be a representation of a compact group G on a finite dimensional vector space V.*

(i) *For every invariant subspace there is an invariant complementary subspace.*

(ii) *The vector space V can be decomposed into a direct sum of irreducible invariant subspaces :*

$$V = V_1 \oplus \cdots \oplus V_N.$$

Preuve. By Proposition there exists on V a Euclidean inner product for which the representation π is unitary. If W is an invariant subspace, then the orthogonal subspace $W^{\perp}$ is invariant and complementary to W. Let V_1 be a non-zero invariant subspace with minimal dimension. Then

$$V = V_1 \oplus V_1^{\perp}$$

■

If $V_1^\perp \neq \{0\}$, let V_2 be a non-zero invariant subspace in $V_1^\perp$ with minimal dimension. One continues the process as long as the subspace $V^\perp$

is not zero. Since the dimension of V is finite, the process stops necessarily.

Théorème 8.1.3 *(Schur's Lemma)* (i) *Let* (π_1, V_1) *and* (π_2, V_2) *be two finite dimensional irreducible representations of a topological group* G*. Let* $A : V_1 \longrightarrow V_2$ *be a linear map which intertwins the representations* π_1 *and* π_2 *:* $A\pi_1(g) = \pi_2(g)A$ *for every* $g \in G$*. Then either* $A = 0$*, or* A *is an isomorphism.*

(ii) *Let* π *be an irreducible* $\mathbb{C}$*-linear representation of a topological group* G *on a finite dimensional complex vector space* V*. Let* $A : V \to V$ *be a* C*-linear map which commutes to the representation* π *:*

$$A\pi(g) = \pi(g)A$$

for every $g \in G$*. Then there exists* $\lambda \in \mathbb{C}$ *such that* $A = \lambda I$*.*

Preuve. (i) In fact, the kernel $\ker(A)$ and the image $Im(A)$ are two invariant subspaces. The statement follows immediately.

(ii) There exists $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is not invertible. It follows from (i) that $A - \lambda I = 0$. If the group G is commutative, by Schur's Lemma an irreducible $\mathbb{C}$-linear representation is one dimensional. It is a character of G. In this setting a character is defined as a continuous function $\chi : G \to C$ satisfying

$$\chi(xy) = \chi(x)\chi(y).$$

For instance the characters of the group $G = SO(2) \simeq U(1) \simeq \mathbb{R}/2\pi\mathbb{Z}$ are the functions

$$\chi^m(\theta) = e^{im\theta} \ (m \in \mathbb{Z}).$$

In part (ii) of Theorem, the assumption that the representation π is $\mathbb{C}$-linear cannot be dropped. For the $\mathbb{R}$-linear representations the situation is quite different. Consider for instance the representation π of the group $G = SO(2) \simeq \mathbb{R}/2\pi\mathbb{Z}$ on $\mathbb{R}^2$ defined by

$$\pi(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

This representation is irreducible. But the matrices :

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

$(a, b \in \mathbb{R})$, commute with the matrices $\pi(\theta)$. (See Exercise 4 about irreducible $\mathbb{R}$-linear representations.) In the same way one can establish similar statements for representations of Lie algebras. ■

Proposition 8.1.4 (i) *Let* (ρ_1, V_1) *and* (ρ_2, V_2) *be two finite dimensional irreducible representations of a Lie algebra* g. *Let* $A : V_1 \longrightarrow V_2$ *be a linear map which intertwins the representations* ρ_1 *and* ρ_2 *:*

$$A\rho_1(X) = \rho_2(X)A$$

for every $X \in g$. *Then either* $A = 0$, *or* A *is an isomorphism.*

(ii) *Let* ρ *be a* $\mathbb{C}$*-linear representation of a complex Lie algebra* g *on a finite dimensional complex vector space* V. *Let* $A : V \longrightarrow V$ *be a* $\mathbb{C}$*-linear map which commutes with the representation* ρ *:*

$$A\rho(X) = \rho(X)A$$

for every $X \in g$. *Then there exists* $\lambda \in \mathbb{C}$ *such that* $A = \lambda I$.

8.2 Compact self-adjoint operators

Let A be a bounded operator on a Hilbert space H. Its norm $\|A\|$ is defined by $\|A\| = \sup_{\|u\| \leq 1} \|Au\|$. For v fixed, the map

$$u \longrightarrow (Au|v)$$

is a continuous linear form on H. By the Riesz representation theorem there exists a unique $w \in H$ such that

$$(Au|v) = (u|w)$$

for every $u \in H$. The map $v \longrightarrow w$ is linear, it is denoted by A^* and is called the adjoint operator of A. It is defined by the relation

$$(Au|v) = (u|A^*v).$$

One can show that $A^* = A$ and that $(A^*)^* = A$. If $A^* = A$, one says that the operator A is self-adjoint, that is $(Au|v) = (u|Av)$ for every $u, v \in H$.

Proposition 8.2.1 *Let A be a self-adjoint operator.*

(i) The eigenvalues of A are real.

(ii) If λ and μ are distinct eigenvalues of A, the corresponding eigenspaces are orthogonal.

Preuve. (a) Let λ be an eigenvalue of A, and u an associated eigenvector :

$$Au = \lambda u, u \neq 0.$$

Then $(Au|u) = (u|Au)$ and $\lambda \|u\|^2 = \overline{\lambda} \|u\|^2$.

(b) Let λ and μ be two eigenvalues of A, $\lambda = \mu$, u and v associated eigenvectors :

$$Au = \lambda u, Av = \mu v.$$

Then

$$(Au|v) = (u|Av)$$

and

$$(\lambda - \mu)(u|v) = 0.$$

■

Proposition 8.2.2 *Let A be a self-adjoint operator. Then* $\|A\| = \sup\limits_{\|u\| \leq 1} |(Au, u|$

Preuve. Put

$$M = \sup_{\|u\| \leq 1} |(Au, u|$$

Observe first that, by the Schwarz inequality, $M \leq A$. On the other hand,

$$\|A\| = \sup_{\|u\| \leq 1} |Re(Au, v|$$

In fact, for $w \in H$,

$$\|w\| = \sup_{\|u\| \leq 1} |(w, v)|,$$

and, by definition of the norm of an operator, and, by definition of the norm of an operator,

$$\|A\| = \sup_{\|u\| \leq 1} \|Au\|$$

From the identity

$$4Re(Au|v) = (A(u+v)|u+v) - (A(u-v)|u-v)$$

it follows that

$$|Re(Au|v)| \leq \frac{M}{4}(\|u+v\|^2 + \|u-v\|^2) = \frac{M}{2}(\|u\|^2 + \|v\|^2).$$

Hence, if $u \leq 1, v \leq 1$,

$$|Re(Au|v)| \leq M,$$

therefore $A \leq M$. Let A be an operator acting on H. The operator A is said to be compact if the following property holds : the image under A of a bounded set is relatively compact. This property is equivalent to each of the two following : the image under A of the unit ball is relatively compact ; if (u_n) is a bounded sequence, there is a subsequence (u_{n_k}) such that the sequence (u_{n_k}) converges. Afinite rank operator is compact. If A is a compact operator and B a bounded operator, then AB and BA are compact operators. ■

Proposition 8.2.3 *If (A_n) is a sequence of compact operators with limit A, $\lim_{n\to\infty} \|A_n - A\| = 0$ then the operator A is compact.*

Preuve. Let (u_k) be a sequence in H such that $u_k \leq 1$. Since the operator A_1 is compact, there is a subsequence $(u_k^{(1)})$ such that the sequence $(A_1 u_k^{(1)})$ converges. Since the operator A_2 is compact, one can extract from the subsequence $(u_k^{(1)})$ a subsequence $(u_k^{(2)})$ such that $(A_2 u_k^{(2)})$ converges, and so on. Then one considers the sequence

$(u_k^{'}) = (u_k^{(2)})$For every n the sequence $k \longmapsto (A_n u_k^{'})$converges. Let us show that $(Au_k^{'})$ is a Cauchy sequence. Let $\varepsilon > 0$. There exists n such that

$$\|A_n - A\| \leq \frac{\varepsilon}{3}$$

Since $(A_n u_k^{'})$ is a Cauchy sequence, there exists $K > 0$ such that, if $k, l \geq K$, $\left\|A_n u_k^{'} - A_n u_l^{'}\right\| \leq \frac{\varepsilon}{3}$, Hence, if $k,\ l \geq K$,

$$\left\|A_n u_k^{'} - A_n u_l^{'}\right\| \leq \left\|Au_k^{'} - A_n u_l^{'}\right\| + \left\|A_n u_k^{'} - A_n u_l^{'}\right\| + \left\|A_n u_l^{'} - Au_l^{'}\right\| \leq \varepsilon$$

Finally we can state that the set of compact operators is a closed two-sided ideal in $L(H)$. ■

Exemple 8.2.4 *Let $H = l^2(N)$. Let (λ_n) be a sequence of complex numbers with limit* 0, *and let $A \in L(H)$ be defined by*

$$A(u_n) = (\lambda_n u_n).$$

The operator A is compact. In fact, let A_N be the operator defined as follows : if $v = A_N u$, $vn = \lambda_n u_n$ if $n \leq N$, $v_n = 0$ if $n > N$. The operator A_N has finite rank and $\|A - A_N\| = \sup\limits_{n>N} |\lambda_n|$

Théorème 8.2.5 *Let A be a compact self-adjoint operator. Then, either A or $-A$ is an eigenvalue of A. Hence, a non-zero compact self-adjoint operator has a non-zero eigenvalue.*

Preuve. Since the operator A is self-adjoint, $\|A\| = \sup\limits_{\|u\|\leq 1} |(Au, v)|$ (Proposition). Observe that the numbers $(Au|u)$ are real ; one may assume, by taking -A instead of A if necessary, that $\|A\| = \sup\limits_{\|u\|\leq 1} |(Au, u)|$. Let us then show that $\lambda = \|A\|$ is an eigenvalue of A. There is a sequence (u_n). such that $\|u_n\| = 1$,

$$\lim_{n\to\infty} (Au_n, u_n) = \lambda$$

Since the operator A is compact, there is a subsequence (u_{n_k}) such that the sequence (Au_{n_k}) converges : $\lim_{k\to\infty} Au_{n_k} = v$. From the expansion

$$\|Au_{n_k} - \lambda u_{n_k}\|^2 = \|Au_{n_k}\|^2 - 2\lambda\,(Au_{n_k}, u_{n_k}) + \lambda^2$$

it follows that

$$\lim_{k\to\infty} \|Au_{n_k} - \lambda u_{n_k}\|^2 = \|v\|^2 - \lambda^2,$$

On the other hand, since $\|A\| = \lambda$, $\lim_{k\to\infty} \|Au_{n_k}\| = \|v\| \leq \lambda$, hence $\lim_{n\to\infty} \|Au_{n_k} - \lambda u_{n_k}\| = 0$ Therefore the sequence (u_{n_k}) converges : $\lim_{k\to\infty} u_{n_k} = u$. Furthermore $Au = v$ and $Au = \lambda u$. ■

Théorème 8.2.6 *(Spectral theorem) Let A be a compact self-adjoint operator. The non-zero eigenvalues of A form a sequence (λ_n) which is finite or converges to 0. Let H_n be the eigenspace associated to λ_n and let P_n be the orthogonal projection onto H_n. The dimension of H_n is finite and*

$$A =_{n=1}^{N} \lambda_n P_n$$

if the number N of non-zero eigenvalues is finite, otherwise

$$A =_{n=0}^{\infty} \lambda_n P_n$$

as a convergent series in the norm topology.

Lemme 8.2.7 *Let H be a Hilbert space. If the unit ball in H is compact, then H is finite dimensional.*

Preuve. If H were not finite dimensional, there would be in H an infinite orthonormal sequence (e_n). Since

$$\|e_p - e_q\| = \sqrt{2}$$

for $p = q$, there cannot be a converging subsequence. Let A be a self-adjoint operator, and λ a non-zero eigenvalue of A. From this lemma it follows that the associated eigenspace is finite dimensional. By Theorem there exists an eigenvalue λ_1 of A such that $|\lambda_1| = \|A\|$. Let H_1 be the associated eigenspace. From Lemma it follows that H1 is finite dimensional. Put $A_1 = A - \lambda_1 P_1$. The operator A is self-adjoint and compact, and $\|A_1\| \leq \|A\|$. By continuing, either one finds an integer N such that $AN = 0$, and then

$$A =_{n=1}^{N} \lambda_n P_n,$$

or the sequence (λ_n) is infinite. Observe that the sequence $(|\lambda_n|)$ is decreasing by construction. Let us show that, when infinite, the sequence (λ_n) goes to 0. Let us assume the opposite, that $|\lambda_n| \geq \alpha > 0$. For every n let us take $v_n \in H_n$, $v_n = 1$. Since A is compact, one can extract from the sequence (Av_n) a converging subsequence. But this is impossible since

$$\|Av_p - Av_q\|^2 = \|\lambda_p v_p - \lambda_q v_q\|^2 = \lambda_p^2 + \lambda_q^2 \geq 2\alpha^2.$$

It follows that

$$A =_{n=1}^{\infty} \lambda_n P_n$$

In fact,

$$A =_{n=1}^{N} \lambda_n P_n + A_{N+1},$$

and $\|A_{N+1}\| = |\lambda_{N+1}|$. Finally, the dimension of Hn is finite since the unit ball of Hn is compact. ■

8.3 Schur orthogonality relations

Let G be a compact group, and μ the normalised Haar measure of G. Let (π, H) be a unitary representation of G. For $v \in H$ one considers the operator K_v of H defined by

$$K_v w = \int_G (w, \pi(g)v)\, \pi(g)v\mu(dg).$$

This can also be written

$$(K_v w, w') = \int_G (w, \pi(g)v)\, \overline{(w', \pi(g)v)}\mu(dg).$$

Proposition 8.3.1 *(i) K_v is bounded, $\|K_v\| \leq \|v\|^2$.*

(ii) K_v is self-adjoint : $K_v^* = K_v$.

(iii) Kv commutes with the representation π : for every $g \in G$, $K_v\pi(g) = \pi(g)K_v$.

(iv) K_v is a compact operator.

Preuve. (i)

$$\|K_v w\| \leq \|v\|^2 \|w\|.$$

(ii)

$$(K_v^* w|w') = (w|K_v w') = (K_v w|w').$$

(iii) Let $g_0 \in G$,

$$K_v\pi(g_0)w = \int_G (w|\pi(g_0^{-1}g)v)\pi(g)v\mu(dg),$$

and, by the invariance of the measure μ,

$$K_v\pi(g_0)w = \int_G (w|\pi(g)v)\pi(g_0 g)v\mu(dg) = \pi(g0)Kvw.$$

(iv) For $v \in H$ let P_v be the rank one operator defined by

$$P_v w = (w|v)v.$$

It is a compact operator and, for v fixed, the map $\begin{array}{l} G \longrightarrow L(H) \\ g \longmapsto P_{\pi(g)v} \end{array}$ is continuous for the norm topology. The operator K_v can be written

$$K_v = \int_G P_{\pi(g)v}\mu(dg)$$

Since the space of compact operators is closed for the norm topology (Proposition), the operator K_v is compact. Observe that

$$(K_v w|w) = \int_G |(\pi(g)v, w)|^2 \mu(dg),$$

and that, if $v = 0$, $(K_v v|v) > 0$, hence $K_v \neq 0$. ∎

Théorème 8.3.2 (*i*) *Every unitary representation of a compact group contains a finite dimensional subrepresentation.*

(*ii*) *Every irreducible unitary representation of a compact group is finite dimensional.*

Preuve. Let (π, H) be a unitary representation of a compact group. The operator K_v is self-adjoint, compact (Proposition), and non-zero if $v \neq 0$. By Theorem it has a non-zero eigenvalue, and the corresponding eigenspace is finite dimensional. This subspace is invariant under the representation π. ■

Théorème 8.3.3 *Let π be an irreducible unitary $\mathbb{C}$-linear representation of a compact group G on a complex Euclidean vector space H with dimension $d\pi$. Then, for $u, v \in H$,*

$$\int_G |(\pi(g)u, v)|^2 \, \mu(dg) = \frac{1}{d_\pi} \|u\|^2 \|v\|^2 ,$$

and, by polarisation, for $u, v, u', v' \in H$,

$$\int_G (w, \pi(g)v) \, \overline{(w', \pi(g)v)} \mu(dg) = \frac{1}{d_\pi} (u, u') (v, v')$$

Preuve. For $v \in H$, the operator K_v commutes with the representation π. By Schur's Lemma (Theorem) there is $\lambda(v) \in \mathbb{C}$ such that

$$K_v = \lambda(v) I.$$

Hence,

$$\int_G |(\pi(g)u, v)|^2 \, \mu(dg) = \lambda(v) \|u\|^2 ,$$

By permuting u and v we get

$$\lambda(u) \|v\|^2 = \lambda(v) \|u\|^2 ,$$

hence

$$\lambda(u) = \lambda_0 \|u\|^2$$

, where λ_0 is a constant. Let $\{e_1, ..., e_n\}$ be an orthonormal basis of

$$H(n = d\pi) :_{i=1}^{n} |\pi(g)u, e_i|^2 = \|u\| ,$$

By integration over G we get

$$\|u\|^2 =_{i=1}^{n} \int_G |(\pi(g)u, e_i)|^2 \mu(dg) = n\lambda_0 \|u\|^2 ,$$

hence $\lambda_0 = 1/n$. Finally

$$\int_G |(\pi(g)u, v)|^2 \mu(dg) = \frac{1}{n} \|u\|^2 \|v\|^2 .$$

Let $\pi_{ij}(g)$ denote the entries of the matrix $\pi(g)$ with respect to the basis $\{e_i\}$,

$$\pi_{ij}(g) = (\pi(g)e_j|e_i)$$

From the preceding theorem one obtains Schur's orthogonality relations :

$$\int_G \pi_{ij}(g)\overline{\pi_{kl}(g)}\mu(dg) = \frac{1}{d_\pi}\delta_{ik}\delta_{jl}.$$

This can be written in the following alternative form : if A and B are two endomorphisms of H, then $\int_G tr(A\pi(g))\overline{tr(B\pi(g)}\mu(dg) = \frac{1}{d_\pi}tr(AB^*)$. In fact one can check that, if A and B are two rank one endomorphisms, the above formula is precisely the second formula of the preceding theorem. Let M_π denote the subspace of $L^2(G)$ generated by the entries of the representation π, that is by the functions of the following form : $g \longrightarrow (\pi(g)u|v)$ $(u, v \in H)$.

■

Théorème 8.3.4 *Let (π, H) and (π', H') be two irreducible unitary representations of a compact group G which are not equivalent. Then M_π and $M_{\pi'}$ are two orthogonal subspaces of $L^2(G)$:*

$$\int_G (v, \pi(g)u)\,\overline{(v', \pi(g)u')}\mu(dg) = 0, (u, v \in H, u', v' \in H').$$

Preuve. Let A be a linear map from H into H' and put

$$\widetilde{A} = \int_G \pi'(g^{-1})A\pi(g)\mu(dg).$$

Then $\widetilde{A}$ is a linear map from H into H' which intertwins the representations π and π'

$$\widetilde{A} \circ \pi(g) = \pi'(g) \circ \widetilde{A}.$$

By Schur's Lemma (Theorem), $\widetilde{A}$. Hence

$$\left(\widetilde{A}u, u'\right) = \int_G (A\pi(g)u, \pi'(g)u')\,\mu(dg) = 0.$$

Take for A the rank one operator defined by

$$Au = (u, v)\,v' \;\; (v \in H, v' \in H'),$$

then $A\pi(g)u = (\pi(g)u|v)v$, and

$$\int_G (v, \pi(g)u)\,\overline{(v', \pi(g)u')}\mu(dg) = 0, (u, v \in H, u', v' \in H').$$

It follows that two irreducible representations π_1 and π_2 of a compact group G are equivalent if and only if the spaces M_{π_1} and M_{π_2} agree. ■

8.4 Peter–Weyl Theorem

Let G be a compact group, and let R denote the right regular representation of G on $L^2(G)$:

$$R(g)f(x) = f(xg).$$

Let (π, H) be an irreducible representation of G, and let $\{e_1, ..., e_n\}$ be an orthonormal basis of H $(n = d_\pi)$. One puts

$$\pi_{ij}(x) = (\pi(x)e_j|e_i).$$

Let $M_\pi^{(1)}$ be the subspace of M_π generated by the entries of the first row, that is by the functions $x \longrightarrow \pi_{1j}(x)$, for $j = 1, ..., n$. Observe that

$$\pi_{1j}(xy) =_{k=1}^{n} \pi_{1k}(x)\pi_{kj}(y).$$

This shows that the subspace $M_\pi^{(1)}$ is invariant under R. Furthermore, the map

$$A :_{j=1}^{n} c_j e_j \longmapsto_{j=1}^{n} c_j \pi_{1j}(x)$$

from H into $M_\pi^{(1)}$ is an isomorphism, and intertwins the representations π and R. In fact, if $u =_{j=1}^{n} c_j e_j$, then

$$\begin{aligned} A\pi(g)u &= A_{j=1}^{n} c_j \pi(g)e_j = A_{j=1}^{n} c_j \left(_{i=1}^{n} \pi_{ij}(g)e_j\right) \\ &= {}_{i=1}^{n} \left(_{j=1}^{n} \pi_{ij}(g)c_j\right) \pi_{1i}(x) = \left(_{j=1}^{n} \pi_{1j}(xg)e_j\right) = R(g)Au \end{aligned}$$

Furthermore $\|Au\|^2 = \frac{1}{n}\|u\|^2$. Let $M_\pi^{(i)}$ denote the subspace of M_π generated by the coefficients of the ithline. Then G

$$M_\pi = M_\pi^{(1)} \oplus \cdots \oplus M_\pi^{(n)},$$

and the restriction to M_π of the representation R is equivalent to $\pi \oplus \cdots \oplus \pi = n\pi$. By considering the columns instead of the rows one gets the same statement with, instead of the representation R, the regular left representation L :

$$L(g)f(x) = f(g^{-1}x).$$

Théorème 8.4.1 *(Peter–Weyl Theorem)*

Let $\widehat{G}$ be the set of equivalence classes of irreducible unitary representations of the compact group G and, for each $\lambda \in \widehat{G}$, let M_λ be the space generated by the coefficients of a representation in the class λ. Then $L^2(G) = \widehat{\oplus_{\lambda\in\widehat{G}}} M_\lambda$. Recall that $\widehat{\oplus_{\lambda\in\widehat{G}}} M_\lambda$ denotes the closure in $L^2(G)$ of $M = \widehat{\oplus_{\lambda\in\widehat{G}}} M_\lambda$ which is the space of finite linear combinations of coefficients of finite dimensional representations of G.

Preuve. We saw that the subspaces M_λ are two by two orthogonal (Theorem). Put

$$\widehat{\oplus_{\lambda\in\widehat{G}}} M_\lambda = H$$

and

$$H_0 = H^\perp.$$

We will show that $H_0 = \{0\}$. Let us assume the opposite, that $H_0 \neq \{0\}$. The spaceH0 is invariant under the representation R and closed. By Theorem it contains a closed subspace $Y \neq \{0\}$ which is invariant under R and irreducible. The restriction of R to Y belongs to one of the classes λ. Let $f \in Y, f \neq 0$, and put

$$F(g) = \int_G f(xg)\overline{f(x)}\mu(dx) = (R(g)f, f).$$

The function F belongs to M_λ. We will see that it is orthogonal to M_λ. Let (π, V) be a representation of the class λ, and $u, v \in V$. Then

$$\int_G F(g)\overline{(\pi(g)u,v)}\mu(dx) = \int_G f(xg)\overline{f(x)(\pi(g)u,v)}\mu(dx)$$

and, by putting $xg = g'$,

$$\int_G F(g)\overline{(\pi(g)u,v)}\mu(dx) = 0$$

Therefore $F = 0$, and, since

$$F(e) = \int_G f(x)\overline{f(x)}\mu(dx) = \int_G |f(x)|^2 \mu(dx) = (R(e)f, f).$$

it follows that $f = 0$. This yields a contradiction. Let H be a finite dimensional Hilbert space and $A \in L(H)$. The Hilbert–Schmidt norm of A is defined by

$$\|A\|^2 = tr(AA^*)$$

If $\{e_1, ..., e_n\}$ is an orthonormal basis of H, and if (a_{ij}) is the matrix of A with respect to this basis, $\|A\| =_{i,j=1}^n |a_{ij}|^2$. For every $\lambda \in \widehat{G}$ one chooses a representative (π_λ, H_λ). Let d_λ denote the dimension of H_λ. If f is an integrable function on G, its Fourier coefficient $\widehat{f}(\lambda)$ is the operator acting on the space H_λ defined by

$$\widehat{f}(\lambda) = \int_G f(g)\pi_\lambda(g^{-1})\mu(dg).$$

The following theorem follows directly from the Peter–Weyl Theorem and from Schur's orthogonality relations. ■

Théorème 8.4.2 *(Plancherel's Theorem) Let $f \in L^2(G)$. Then f is equal to the sum of its Fourier series :*

(i)

$$f(g) =_{\lambda\in\widehat{G}} d_\lambda tr(\widehat{f}(\lambda)\pi_\lambda(g).$$

This holds in the L^2 sense.

(ii)

$$\int_G |f(x)|^2 \mu(dx) =_{\lambda\in\widehat{G}} d_\lambda \left\|(\widehat{f}(\lambda)\right\|^2$$

And, if $f_1, f_2 \in L^2(G)$,

$$\int_G f_1(x)\overline{f_2(x)}\mu(dx) =_{\lambda\in\widehat{G}} d_\lambda tr(\widehat{f_1}(\lambda)(f_2(\lambda))^*.$$

(iii) The map $f \longmapsto \widehat{f}$ *is a unitary isomorphism from* $L^2(G)$ *onto the space of sequences of operators* $A = (A_\lambda)$ $(A_\lambda \in L(H_\lambda))$, *for which*

$$\|A\|^2 =_{\lambda\in\widehat{G}} d_\lambda \|A_\lambda\|^2,$$

and equipped with this norm. If the compact group G *is commutative then a* $\mathbb{C}$*-linear irreducible representation is one dimensional, and* $\widehat{G}$ *is the set of continuous characters. Recall that, in this setting, a continuous character is a continuous function*

$$\chi : G \longrightarrow \mathbb{C}^*$$

satisfying

$$\chi(xy) = \chi(x)\chi(y).$$

Since G *is compact, the set* $\chi(G)$ *is a compact subgroup of* $\mathbb{C}^*$, *hence consists of modulus one complex numbers. Therefore* $\chi : G \to \{z \in \mathbb{C} \text{ tel que } |z| = 1\}$. *The set* $\widehat{G}$ *is a commutative group for the ordinary product of the characters which is called the dual group of* G, *and the continuous characters form a Hilbert basis of* $L^2(G)$.*The Fourier coefficient* $\widehat{f}(\chi)$ *of a square integrable function* f *on* G *is given by* $\widehat{f}(\chi) = \int_G f(x)\overline{\chi(x)}\mu(dx)$ *The Fourier series of* f *is written as :*

$$_{\chi\in\widehat{G}}\widehat{f}(\chi)\chi(x)$$

and the Plancherel formula :

$$\int_G |f(x)|^2 \mu(dx) =_{\chi\in\widehat{G}} \left|\widehat{f}(\chi)\right|^2.$$

For instance, if $G = SO(2) \simeq U(1) \simeq \mathbb{R}/2\pi\mathbb{Z}$, *then a character* χ *has the form*

$$\chi(\theta) = e^{im\theta},$$

where $m \in \mathbb{Z}$. Hence $\widehat{G} = \mathbb{Z}$. In this case one obtains the classical formulae. If f is an integrable function on $\mathbb{R}/2\pi\mathbb{Z}$, the Fourier coefficients of f are given by

$$\widehat{f}(m) = \frac{1}{2\pi}\int_G f(\theta)e^{-im\theta}\mu(d\theta).$$

The Fourier series of f is written as ,

$$_{m\in\mathbb{Z}}\widehat{f}(m)e^{im\theta}$$

and the Plancherel formula, if f is square integrable, $\frac{1}{2\pi}\int_G |f(\theta)|^2 d\theta =_{m\in\mathbb{Z}} \left|\widehat{f}(m)\right|^2$. Recall thatMdenotes the space of finite linear combinations of coefficients of finite dimensional representations of G, $M =_{\lambda\in\widehat{G}} M_\lambda$ We will show that M is dense in the space of continuous complex valued functions on G. For that we will apply the Stone–Weierstrass Theorem which we recall below.

Théorème 8.4.3 *(Stone–Weierstrass Theorem) Let X be a compact topological space, and $C(X)$ the space of complex valued continuous functions on X, equipped with the topology of uniform convergence. Let A be a subspace of $C(X)$ with the following properties :*

(i) A is an algebra (for the ordinary product of functions),

(ii) A separates the points of X, and constant functions belong to A,

(iii) if f belongs to A, then ¯f also belongs to A.

Then A is dense in $C(X)$. See for instance : K. Yosida (1968). *Functional Analysis. Springer (Corollary). Let (π_1, H_1) and (π_2, H_2) be two finite dimensional representations of G. The tensor product $\pi_1 \otimes \pi_2$ is the representation of G on $H_1 \otimes H_2$ such that*

$$(\pi_1 \otimes \pi_2)(g)(u_1 \otimes u_2) = \pi_1(g)u_1 \otimes \pi_2(g)u_2.$$

If H_1 and H_2 are finite dimensional Hilbert spaces, then $H_1 \otimes H_2$ is equipped with an inner product such that

$$(u_1 \otimes u_2|v_1 \otimes v_2) = (u_1|v_1)(u_2|v_2),$$

and

$$(\pi_1 \otimes \pi_2)(g)(u_1 \otimes u_2)|(v_1 \otimes v_2) = (\pi_1(g)u_1|v_1)(\pi_2(g)u_2|v_2).$$

Therefore the product of a coefficient of π_1 *and a coefficient of* π_2 *is a coefficient of* $\pi_1 \otimes \pi_2$. *For* $\lambda, \mu \in \widehat{G}$ *the representation* $\pi_\lambda \otimes \pi_\mu$ *can be decomposed into a sum of irreducible representations :*

$$\pi_\lambda \otimes \pi_\mu =_{v\in E(\lambda,\mu)} c(\lambda, \mu; v)\pi_v,$$

, where $E(\lambda, \mu)$ *is a finite subset of* $\widehat{G}$. *The numbers* $c(\lambda, \mu; v)$, *which are positive integers, are called Clebsch–Gordan coefficients. This shows that the space* A *of finite linear combinations of coefficients of finite dimensional representations of* G *is an algebra. Let* V *be a normed vector space, and* V *its topological dual. Let* π *be a representation of* G *on* V. *The contragredient representation of* π *is the representation* π *of* G *on* V *defined by*

$$(\pi'(g)f, u) = (f, \pi(g - 1)u) \ \ (f \in V', u \in V).$$

Assume now that $V = H$ *is a Hilbert space and that* π *is unitary. There is an antilinear isomorphism* T *from* H *onto* H' *defined by*

$$(Tv, u) = (u|v).$$

Lemme 8.4.4 *Let* G *be a compact group. If* $g \neq e$, *there exists a finite dimensional representation* π *of* G *such that* $\pi(g) \neq I$.

8.5 Characters and central functions

Let G be a compact group. A function f which is defined on G is said to be central if

$$f(gxg^{-1}) = f(x) \ (g, x \in G).$$

Let π be a representation of G on a finite dimensional complex vector space V. The character of π is the function χ_π defined on G by

$$\chi_\pi(g) = tr\pi(g).$$

It is a central function which only depends on the equivalence class of π. One can establish easily the following properties :

$$\begin{aligned}
\chi_\pi(e) &= dimV, \\
\chi_{\pi_1\oplus\pi_2}(g) &= \chi_{\pi_1}(g) + \chi_{\pi_2}(g), \\
\chi_{\pi_1\oplus\pi_2}(g) &= \chi_{\pi_1}(g)\chi_{\pi_2}(g), \\
\chi_{\overline{\pi}}(g) &= \chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}.
\end{aligned}$$

Let us denote by V^G the subspace of invariant vectors :

$$V^G = \{v \in V | \forall g \in G, \pi(g)v = v\}.$$

The operator P, defined by

$$Pv = \int_G \pi(g)v\mu(dg),$$

where μ is the normalised Haar measure of G, is a projection onto V^G. Since

$$trP = dimV^G,$$

it follows that

$$\int_G \chi_\pi(g)d\mu(g) = dimV^G$$

. If (π_1, V_1) and (π_2, V_2) are two finite dimensional representations of G one puts

$$E(\pi_1, \pi_2) = \{A \in L(V_1, V_2) | \forall g \in G, A\pi_1(g) = \pi_2(g)A\}.$$

This is the space of operators which intertwin the representations π_1 and π_2. The group G acts on the space $L(V_1, V_2)$ by the representation T defined by

$$T(g)A = \pi_2(g)A\pi_1(g^{-1}).$$

This representation is equivalent to $\pi_2 \otimes \overline{\pi_1}$. Observe that

$$E(\pi_1, \pi_2) = L(V_1, V_2)^G,$$

Proposition 8.5.1 *Let π_1 and π_2 be two finite dimensional representations of G. Then*

$$\int_G \chi_{\pi_1}(g)\overline{\chi_{\pi_2}(g)}\mu(dg) = dimE(\pi_1, \pi_2).$$

Assume that π_1 and π_2 are irreducible. They are equivalent if and only if they have the same character :

$$\chi_{\pi_1}(g) = \chi_{\pi_2}(g) \ (g \in G)$$

A finite dimensional representation π of G is irreducible if and only if

$$\int_G |\chi_\pi(g)|^2\mu(dg) = 1.$$

If π is an irreducible representation of the compact group G, then

$$\int_G \chi_\pi(xgyg^{-1})\mu(dg) = \frac{1}{d\pi}\chi_\pi(x)\chi_\pi(y).$$

Chapitre 9

Induced Representations

9.1 The Definition

Let L be a unitary representation of H on a Hilbert space E.

Définition 9.1.1 *Let $\mathcal{E}^L$ be the linear space of all functions f from G to E such that :*

1. *f is d_G-measurable ;*
2. *$f(x\xi) = (\delta_H(\xi)/\delta_G(\xi))^{1/2} L(\xi^{-1}) f(x)$ whenever $\xi \in H$, $x \in G$;*
3. *$\|f(\cdot)\|^2$ is locally summable on G.*

It is clear that $x \mapsto \|f(x)\|^2$ is d_G-measurable ($f \in \mathcal{E}^L$) ; moreover, the polarization identity together with (3) imply that $x \mapsto (f(x), g(x))$ is d_G-measurable and locally summable ($f, g \in \mathcal{E}^L$).

Given $\phi \in C_c(G)$, set $\dot{\phi}(\dot{x}) = \int_H \phi(x\xi) d_H(\xi)\ \dot{x}\ (\dot{x} = xH)-$ then the assignment $\phi \mapsto \dot{\phi}$ is a continuous surjection of $C_c(G)$ onto $C_c(G/H)$ (cf. infra).

Lemme 9.1.2 *Let $f, g \in \mathcal{E}^L-$ then we may define a Radon measure $\mu_{f,g}$ on G/H by the rule :*

$$\int_G (f(x), g(x))\, \phi(x) d_G(x) = \int_{G/H} \dot{\phi}(\dot{x})\, d\mu_{f,g}(\dot{x}) \quad (\phi \in C_c(G)).$$

Preuve. To verify the lemma, one need only prove that

$$\dot{\phi} = 0 \Rightarrow \int_G (f(x), g(x))\, \phi(x) d_G(x) = 0.$$

For this purpose, fix an arbitrary $\psi \in C_c(G)$.
Noting that $\dot{\phi} = 0 \Rightarrow$

$$\int_H \phi(x\xi) d_H(\xi) = \int_H \phi\left(x\xi^{-1}\right) \delta_H\left(\xi^{-1}\right) d_H(\xi) = 0$$

we have

$$\begin{aligned} 0 &= \int_G \int_H (f(x), g(x))\, \psi(x) \phi\left(x\xi^{-1}\right) \delta_H\left(\xi^{-1}\right) d_H(\xi) d_G(x) \\ &= \int_H \int_G (f(x), g(x))\, \psi(x) \phi\left(x\xi^{-1}\right) \delta_H\left(\xi^{-1}\right) d_G(x) d_H(\xi) \\ &= \int_H \int_G (f(x\xi), g(x\xi))\, \psi(x\xi) \phi(x) \delta_G(\xi) \delta_H\left(\xi^{-1}\right) d_G(x) d_H(\xi) \\ &= \int_G (f(x), g(x))\, \phi(x) \left[\int_H \psi(x\xi) d_H(\xi)\right] d_G(x). \end{aligned}$$

All that remains to do is choose ψ so that the function $x \mapsto \int_H \psi\,(x\xi)\, d_H(\xi)$ is 1 on the compact support of ϕ. □

Définition 9.1.3 *Let E^L denote the subset of $\mathcal{E}^L$ consisting of those f such that $\mu_{f,f}(G/H) < \infty$.*

Due to the fact that

$$\|f(\cdot) + g(\cdot)\|^2 \leqslant \|f(\cdot)\|^2 + \|g(\cdot)\|^2 \qquad \left(f, g \in \mathcal{E}^L\right),$$

it is clear that E^L is a linear subspace of $\mathcal{E}^L$. Evidently $f, g \in E^L \Rightarrow \mu_{f,g}(G/H) < \infty$. Set $(f, g) = \mu_{f,g}(G/H) \quad \left(f, g \in E^L\right)$; plainly $(\cdot,\cdot)$ is a positive semi-definite Hermitian form on E^L. Agreeing to identify functions which are equal almost everywhere, we then see that the form $(\cdot,\cdot)$ equips E^L with the structure of a pre-Hilbert space. In order to show completeness, we need the following estimate. Write $\|f\|$ for $(f, f)^{1/2} \quad \left(f \in E^L\right)$.

Lemme 9.1.4 *For each compact subset ω of G, there exists a constant r_ω such that for all $f \in E^L$,*

$$\int_\omega \|f(x)\| d_G(x) \leqslant r_\omega \|f\|.$$

Preuve. Choose $\phi \in C_c^+(G)$ such that $\phi = 1$ on $\omega-$ then

$$\int_\omega \|f(x)\|^2 d_G(x) \leqslant \int_G \phi(x)\|f(x)\|^2 d_G(x) = \mu_{f,f}\left(\dot{\phi}\right) \leqslant \left\|\dot{\phi}\right\|_\infty \|f\|^2.$$

Therefore, thanks to Schwarz inequality, we may take

$$r_\omega = \left(\left\|\dot{\phi}\right\|_\infty \int_\omega d_G(x)\right)^{1/2}. \square$$

Proposition 9.1.5 *The space E^L is complete.*

Preuve. Let $\{f_n\}$ be a Cauchy sequence in E^L ; in order to show that $\{f_n\}$ has a limit in E^L we may, by passing to a subsequence if necessary, assume that $\|f_n - f_{n+1}\| < 2^{-n}$.

(1) We begin by proving that $\lim_{n\to\infty} f_n(x)$ exists in E for almost all $x \in G$. Thus let ω be a compact subset of $G-$ then, owing to Lemma 9.1.4, we have

$$\int_\omega \|f_n(x) - f_{n+1}(x)\| d_G(x) < 2^{-n} r_\omega \Rightarrow \int_\omega \left(\sum_1^\infty \|f_n(x) - f_{n+1}(x)\|\right) d_G(x) < r_\omega.$$

Therefore, for almost all $x \in \omega$, $\{f_n(x)\}$ is Cauchy in E. Put $f(x) = \lim_{n\to\infty} f_n(x)$ or 0 according to whether the limit exists or not. Of course f is d_G-measurable and verifies, almost everywhere,

$$f(x\xi) = \left(\frac{\delta_H(\xi)}{\delta_G(\xi)}\right)^{1/2} L\left(\xi^{-1}\right) f(x) \qquad (\xi \in H, x \in G).$$

(2) We still have to show that $\|f(\cdot)\|^2$ is locally summable, that $\|f\| < \infty$, and that $\|f_n - f\| \to 0$. Let $\phi \in C_c^+(G)-$ then, iterating the parallelogram identity for E, we find that

$$\begin{aligned}\int_G \|f_n(x) - f_{n+m}\|^2 \phi(x) d_G(x) &\leqslant \sum_{i=1}^\infty 2^i \int_G \|f_{n+i-1}(x) - f_{n+i}(x)\|^2 \phi(x) d_G(x) \\ &\leqslant \sum_{i=1}^\infty 2^i \|f_{n+i-1} - f_{n+i}\|^2 \left\|\dot{\phi}\right\|_\infty \\ &< 2^{-2n+2} \left\|\dot{\phi}\right\|_\infty.\end{aligned}$$

By Fatou's Lemma, $\int_G \|f_n(x) - f(x)\|^2 \phi(x) d_G(x) \leqslant 2^{-2n+2} \left\|\dot{\phi}\right\|_\infty$. Letting ω be compact in G and taking $\phi = 1$ on ω, we see that $\|f_n(\cdot) - f(\cdot)\|^2$ is summable on ω, whence $f_n - f \in \mathcal{E}^L \Rightarrow f \in \mathcal{E}^L$. On the other hand, as ϕ was arbitrary in $G_c^+(G)$, our computation also tells us that

$$\|f_n - f\| \leqslant 2^{-2n+2} \Rightarrow f \in E^L;$$

finally, it is now obvious that $f_n \to f$ in E^L.

The proposition is thereby established. □

A priori, our Hilbert space E^L might consist of zero alone ; to show that this is not the case, we shall now discuss a procedure, introduced by Mackey and elaborated upon by Bruhat, which serve to establish among other things, that E^L contains plenty of functions .

Convention Let E and F be two Hausdorff topological vector spaces over $\mathbb{C}$, f a linear map of E onto F. We shall say that f is a *strict morphism* if the canonical bijection of $E/f^{-1}(0)$ onto F which is associated with f is an isomorphism (of topological vector spaces). For this, it is necessary and sufficient that f be continuous and open.

It will be convenient to make a temporary change in our hypotheses ; thus, let E be a Fréchet space, L a differentiable representation of H on $E-$consider the space ${}^LC^\infty(G; E)$ of all functions f on G with values in E such that :

1. [(a)]
2. $f(x\xi) = \rho_H(\xi)^{1/2} L(\xi^{-1}) f(x)$ for all $\xi \in H$ and $x \in G$;
3. The canonical image in G/H of the support of f is compact (briefly f has compact support mod H) ;
4. $f \in C^\infty(G; E)$.
   [Here ρ_H is to be taken as in appendix 1 ; in particular, then, $\rho_H(\xi) = (\delta_H(\xi)/\delta_G(\xi))$ $(\xi \in H)$.]
   The space ${}^LC^\infty(G; E)$ may be topologized in the following way. Let ω be a compact subset of G ; let ${}^LC^\infty_\omega(G; E)$ be the subspace of ${}^LC^\infty(G; E)$

comprised of those functions with support contained in ωH. Place on ${}^L C_\omega^\infty(G;E)$ the ralative topology inherited from $C^\infty(G;E)$; in this topology ${}^L C_\omega^\infty(G;E)$ is a Fréchet space.
Equip ${}^L C_\omega^\infty(G;E)$ with the strict inductive limit of the topologies of the ${}^L C_\omega^\infty(G;E)-$ then ${}^L C^\infty(G;E)$ is an $LF-$space.
Given $f \in C_c^\infty(G;E)$, put

$$f^L(x) = \int_H \rho_H(\xi)^{-1/2} L(\xi) f(x\xi) d_H(\xi) \quad (x \in G; d_H \text{ left Haar measure on } H).$$

[The integral appearing on the right here exist in the Bochner sense, since the integrand is continuous and has compact support.]

Lemme 9.1.6 (Bruhat) *The map π^L, $f \mapsto f^L$, is a continuous surjection of $C_c^\infty(G;E)$ onto ${}^L C^\infty(G;E)-$ in fact, it is a strict morphism.*

Preuve. (1) Plainly f^L satisfies (a) and (b) above. As for (c) observe that for each $X \in g$ (viewed as a right invariant differential operator on G),

$$X f^L(x) = \lim_{t\to 0} \int_H \rho_H(\xi)^{-1/2} L(\xi) \left[\frac{f(\exp(-tX)x\xi) - f(x\xi)}{t}\right] d_H(\xi) \quad (x \in G).$$

Since $\operatorname{spt}(f)$ is compact and since the operators $L(\xi)$ constitute an equicontinuous set when ξ runs through a compact subset of H, we see that the limit on the right hand side, as $t \to 0$, exists. Hence $f^L \in C^\infty(G;E)$ and for each right invariant differential operator $D \in \mathfrak{G}$ we have

$$D f^L(x) = \int_H \rho_H(\xi)^{-1/2} L(\xi) D f(x\xi) d_H(\xi) \quad (x \in G).$$

In addition note that $f \in C_\omega^\infty(G;E) \Rightarrow f^L \in^L C_\omega^\infty(G;E) \quad (\omega$ a compact subset of $G)$.

(2) Let us now establish the continuity of the map $f \mapsto f^L$. Il suffices to show that the map $f \mapsto f^L$ of $C_\omega^\infty(G;E)$ into ${}^L C_\omega^\infty(G;E)$ is continuous (ω a compact subset of G). So suppose $f_n \to 0$ in $C_\omega^\infty(G;E)$. We must prove that $D f_n^L \to 0$ on each compact subset $\omega_1 \subset G$ $(D \in \mathfrak{G})$. Let $\mathcal{P}_1$ be a closed convex balanced neighborhood of zero in E ; per $\mathcal{P}_1$, select a neighborhood $\mathcal{P}_2$ of zero in E such that $a \in \mathcal{P}_2, \xi \in \omega_1^{-1}\omega \cap H \Rightarrow L(\xi)a \in \mathcal{P}_1$.

$\Big[$Evidently the function $\xi \mapsto f(x\xi) \quad (f \in C_\omega^\infty(G;E))$ has its support in $\in \omega_1^{-1}\omega \cap H$ if $x \in \omega_1$.$\Big]$ Now we may find an index n_0 so that $n \geqslant n_0 \Rightarrow Df_n(x) \in \mathcal{P}_2$ for all $x \in G$; therefore, if $n \geqslant n_0$, then

$$Df_n^L(x) \in \left(\int_{\omega_2} \rho_H(\xi)^{-1/2} d_H(\xi)\right) \mathcal{P}_1 \quad (\omega_2 = \omega_1^{-1}\omega \cap H)$$

for all $x \in \omega_1$ and so the continuity of the map $f \mapsto f^L$ follows.

(3) We have yet to prove that $f \mapsto f^L$ is surjective ; in so doing it will be seen that our mapping is actually a strict morphism. First of all observe that if $\alpha \in C_c^\infty(G/H)$ and $h \in^L C^\infty(G;E)$, then the function $x \mapsto \alpha(\dot{x})\, h(x)$ is still in ${}^LC^\infty(G;E)$ (the map $x \mapsto \pi(x) = \dot{x}$ denoting, as usual, the canonical projection of G onto G/H). Hence ${}^LC^\infty(G;E)$ is a module over $C_c^\infty(G/H)$. Viewing G in the usual way as a principal fiber bundle over G/H, select a covering $\{\mathcal{O}_i\}$ of G/H by open sets in each of which there exists a C^∞ section $\dot{x} \mapsto s_i(\dot{x})$ of $\pi^{-1}(\mathcal{O}_\rangle)$ fibered by H. Let $\{\phi_i\}$ be a C^∞ partition of unity subordinate to this covering and let ψ be a function in $C_c^\infty(H)$ such that $\int_H \rho_H(\xi)^{-1/2}\psi(\xi) d_H(\xi) = 1$. For $h \in^L C^\infty(G;E)$, put

$${}_Lh(x) = \sum_i \phi_i(\dot{x})\, \psi\left(s_i(\dot{x})^{-1} x\right) L\left(x^{-1} s_i(\dot{x})\right) h\left(s_i(\dot{x})\right) \quad (x \in G).$$

Then ${}_Lh \in C_c^\infty(G;E), h \mapsto_L h$ is continuous map from ${}^LC^\infty(G;E)$ into $C_c^\infty(G;E)$ and ${}_Lh^L = h$, which shows that $h \mapsto_L h$ is a continuous right inverse to $f \mapsto f^L$. Therefore the map $f \mapsto f^L$ of $C_c^\infty(G;E)$ into ${}^LC^\infty(G;E)$ is indeed a strict morphism.

Hence the lemma. □

Remarks (1) The proof of lemma 9.1.6 shows that the map $f \mapsto f^L$ of $C_c^p(G;E)$ into the space ${}^LC_c^p(G;E)$ of p times $(0 \leqslant p \leqslant \infty)$ differentiable functions on G with values in E verifying (a) and (b) above, equipped with the evident topology, is a strict morphism. Take in particular for L the representation $\xi \mapsto \rho_H(\xi)^{1/2}$ $(\dim(E) = 1)$ - then ${}^LC_c^p(G)$ is canonically identified with the space $C_c^p(G/H)$, and thus the map $f \mapsto f^L$ of $C_c^p(G)$ into $C_c^p(G/H)$ is a strict morphism, a well-known result due to Weil[1].

(2) Let us suppose that L is merely a continuous representation of H on our Fréchet space E ; then, of course, the natural counterpart to the space

${}^{L}C^{\infty}(G;E)$ (in the differentiable case) is the space ${}^{L}C^{0}(G;E) =^{L} C(G;E)$ of continuous E-valued functions f on G, having compact support modulo H and verifying the relation

$$f(x\xi) = \rho_H(\xi)^{1/2} L\left(\xi^{-1}\right) f(x) \quad (\xi \in H, x \in G).$$

Equipping the spaces $C_c(G;E)$, ${}^{L}C(G;E)$ with the evident inductive limit topologies, one verifies without difficulty that the map

$$\pi^L, f \mapsto f^L, f^L(x) = \int_H \rho_H(\xi)^{-1/2} L(\xi) f(x\xi) d_H(\xi) \quad (x \in G),$$

is a strict morphism of $C_c(G;E)$ onto ${}^{L}C(G;E)$ (cf. Remark 1).

We shall now return to the situation initially under consideration ; thus, in particular, we are once again working with a unitary representation L of H on a Hilbert space E. As in number 4.4.1, let L_∞ be the differentiable representation of H on E_∞ canonically associated with L. As we know, E_∞ is a Fréchet space ; hence it makes sense to consider the space ${}^{L_\infty}C^{\infty}(G;E_\infty)-$ plainly ${}^{L_\infty}C^{\infty}(G;E_\infty) \subset^{L} C(G;E) \subset E^L$. [Let us recall that the topology on E_∞ is not the relative topology induced by E but rather the (in general finer) topology which E_∞ inherits from $C^{\infty}(H;E)$ via the map $a \mapsto \tilde{a}, \tilde{a}(\xi) = L(\xi)a \quad (a \in E_\infty, \xi \in H)$.]

9.2 The carrier space of the induced representation

Lemme 9.2.1 *The injection of ${}^{L_\infty}C^{\infty}(G;E_\infty)$ into E^L is continuous and, in fact, admits a continuous extension to ${}^{L}C(G;E)-$ furthermore, ${}^{L_\infty}C^{\infty}(G;E_\infty)$ is dense in E^L.*

Preuve. (1) Let us begin by estimating $\left\| f^{L_\infty} \right\| \quad (f \in C_c^{\infty}(G;E_\infty))$. Choose $\phi \in C_c^{+}(G)$ such that $\dot{\phi} = 1$ on $\dot{spt}(f)$. Let $g \in E^L$. Since f^{L_∞} vanishes outside of $\omega H (\omega = spt(f))$, the measure $\mu_{f L_\infty, g}$ must be carried by $\dot{spt}(f)$. Hence, using Fubini's Theorem, we find that

$$\begin{aligned}
(f^{L_\infty}, g) &= \mu_{f^{L_\infty}, g}(G/H) \\
&= \int_G \phi(x)\left(f^{L_\infty}(x), g(x)\right) d_G(x) \\
&= \int_G \int_H \phi(x)\delta_G(\xi)^{1/2}\delta_H(\xi)^{-1/2}(f(x\xi), L(\xi^{-1})g(x))d_H(\xi)d_G(x) \\
&= \int_G \int_H \phi(x)\delta_G(\xi)\delta_H(\xi)^{-1}(f(x\xi), g(x\xi))d_H(\xi)d_G(x) \\
&= \int_G \int_H \phi(x)\delta_G\left(\xi^{-1}\right)\left(f\left(x\xi^{-1}\right), g\left(x\xi^{-1}\right)\right) d_H(\xi)d_G(x) \\
&= \int_H \int_G \phi(x\xi)(f(x), g(x))d_G(x)d_H(\xi) \\
&= \int_G (f(x), g(x))d_G(x).
\end{aligned}$$

Conclusion :

$$\left|(f^{L_\infty}, g)\right| \leqslant r_{spt(f)}\|f\|_\infty\|g\| \Rightarrow \left\|f^{L_\infty}\right\| \leqslant r_{spt(f)}\|f\|_\infty$$

(cf. Lemma 9.1.4).

(2) The estimate in (1), together whith Lemma 9.1.6, implies that the injection of ${}^{L_\infty}C^\infty(G; E_\infty)$ into E^L is continuous ; the argument also shows that this injection admits a continuous extension to ${}^LC(G; E)$. [As was shown in part (3) of the proof of Lemma 9.1.6, the map $f \mapsto f^{L_\infty}$ of $C_c^\infty(G; E_\infty)$ onto ${}^{L_\infty}C^\infty(G; E_\infty)$ admits a continuous right inverse $h \mapsto_{L_\infty} h \quad \left(h \in^{L_\infty} C^\infty(G; E_\infty)\right)$. So suppose, for instance, that $h_n \to 0$ in ${}^{L_\infty}C^\infty(G; E_\infty)$; to verify that $h_n \to 0$ in E^L, first note that ${}_{L_\infty}h_n \to 0$ in $C_c^\infty(G; E_\infty)$. This being the case, select a compact set $\omega \subset G$ such that $spt\,({}_{L_\infty}h_n) \subset \omega$ (all n) ; in view of preceding estimate, we then have that

$$\|{}_{L_\infty}h_n^{L_\infty}\| = \|h_n\| \leqslant r_\omega\|{}_{L_\infty}h_n\|_\infty \to 0$$

as $n \to \infty$.]

(3) Let $f \in C_c^\infty(G), a \in E_\infty-$ then

$$\left(f \bigotimes a\right)^{L_\infty}(x) = \int_H \rho_H(\xi)^{-1/2}f(x\xi)L_\infty(\xi)ad_H(\xi) \quad (x \in G).$$

To prove that ${}^{L_\infty}C^\infty(G; E_\infty))$ is dense in E^L, it suffices to show that the $(f \bigotimes a)^{L_\infty}$

$(f \in C_c^\infty(G), a \in E_\infty)$ span a dense subspace of E^L (since E^L is complete) ; in turn, for this, it will be enough to show that only zero is orthogonal to all the $(f \bigotimes a)^{L_\infty}$.

If $\Big((f \bigotimes a)^{L_\infty}, g\Big) = 0$ for some $g \in E^L$ (all $f \in C_c^\infty(G), a \in E_\infty$), then, by (1) above,

$$\int_G f(x)(a, g(x))d_G(x) = 0 \quad (f \in C_c^\infty(G), a \in E_\infty).$$

Hence $(a, g(x)) = 0$ almost everywhere on G (all $a \in E_\infty$). Therefore $g(x) = 0$ almost everywhere on G.

The proof of Lemma 9.2.1 is now complete. □

With these preliminaries out of the way, let us proceed to the definition of the unitary representation U^L of G induced by the unitary representation L of H on E.

The reprentation space for U^L will be the Hilbert space E^L, given f ∈E^L, U^L $(x)\, f = f \circ L_{x^{-1}}$ i.e the left translation of f by x^{-1} (x∈ G).

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